

## Introduction to Sturm-Liouville Theory and the Theory of Generalized Fourier Series

We start with some introductory examples.

### 5.1. Cauchy's equation

The homogeneous Euler-Cauchy equation ([Leonhard Euler](#) and [Augustin-Louis Cauchy](#)) is a linear homogeneous ODE which can be written as

$$(*) \quad x^2 y'' + axy' + by = 0.$$

**Example 5.1.** Solve the equation (\*).

**Solution:** Set  $y(x) = x^r$ , then  $y'(x) = rx^{r-1}$  and  $y''(x) = r(r-1)x^{r-2}$ . If we insert this into (\*) we get

$$r(r-1)x^r + arx^r + bx^r = 0,$$

which gives us the equation

$$(**) \quad \boxed{r(r-1) + ar + b = 0.}$$

This is the so-called *Characteristic equation* corresponding to (\*). Assume that the solutions of (\*\*) are  $r_1$  and  $r_2$ . We have three different cases:

1. If  $r_1$  and  $r_2$  are real and different,  $r_1 \neq r_2$  then
 
$$y(x) = Ax^{r_1} + Bx^{r_2}.$$
2. If  $r_1$  and  $r_2$  are real and equal,  $r_1 = r_2 = r$  then
 
$$y(x) = Ax^r + Bx^r \ln x.$$
3. If  $r_1$  and  $r_2$  are complex conjugates,  $r_1 = \alpha + i\beta$ ,  $r_2 = \alpha - i\beta$  then
 
$$y(x) = Ax^{\alpha+i\beta} + Bx^{\alpha-i\beta}.$$

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REMARK 6. Observe that

$$x^{\alpha+i\beta} = x^\alpha e^{i\beta \ln x} = x^\alpha (\cos(\beta \ln x) + i \sin(\beta \ln x))$$

and

$$x^{\alpha-i\beta} = x^\alpha (\cos(\beta \ln x) - i \sin(\beta \ln x)),$$

hence we can write the solution of case 3 in the example above as

$$y(x) = x^\alpha ((A+B) \cos(\beta \ln x) + i(A-B) \sin(\beta \ln x)).$$

If we only consider constants  $A$  and  $B$  such that  $C = A + B$  and  $D = i(A - B)$  are real numbers then we see that

$$y(x) = x^\alpha (C \cos(\beta \ln x) + D \sin(\beta \ln x))$$

is a *real-valued solution* to (\*).

**Example 5.2.** Solve the differential equation

$$x^2 y'' + 2xy' - 6y = 0.$$

**Solution:** The characteristic equation is

$$r(r-1) + 2r - 6 = 0,$$

i.e.

$$r^2 + r - 6 = 0$$

which has the solutions

$$r_1 = 2, r_2 = -3.$$

Since we have two different real solutions we are in case 1 above, and the general solution to the differential equation is given by

$$y(x) = Ax^2 + Bx^{-3}.$$

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**Example 5.3.** Solve the equation

$$x^2 y'' + 2xy' + \lambda y = 0, \lambda > \frac{1}{4}.$$

**Solution:** The characteristic equation is

$$r^2 + r + \lambda = 0,$$

with solutions

$$r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}.$$

Since we now have two complex conjugate solutions  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$  we are in case 3 above, and the general solution to the differential equation is given by

$$y(x) = Ax^{-\frac{1}{2}} \left( \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln x \right) + B \cos \left( \sqrt{\lambda - \frac{1}{4}} \ln x \right) \right).$$

## 5.2. Examples of Sturm-Liouville Problems

In the next section we will describe in more details what is meant by a Sturm-Liouville problem ([Charles-Francois Sturm](#) and [Joseph Liouville](#)), but first we will look at some examples.

**Example 5.4.** Solve

$$\begin{cases} y'' + \lambda y = 0, \\ y(0) = y(l) = 0. \end{cases}$$

**Solution:** Previously (cf. section 4.8, p. 22) we saw that this problem can be solved if and only if

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, n = 1, 2, 3, \dots \text{(eigenvalues)}$$

with the corresponding solutions

$$y_n(x) = a_n \sin\left(\frac{n\pi}{l}x\right) \text{ (eigenfunctions).}$$

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**Example 5.5.** Solve

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 \leq x \leq 1, \\ X(0) = 0, \\ X'(1) = -3X(1). \end{cases}$$

**Solution:** We have three different cases:

$\lambda = 0$   $X(x) = Ax + B$ ,  $X(0) = 0 \Rightarrow B = 0$ , and  $X'(1) = -3X(1) \Rightarrow A = -3A \Rightarrow A = 0$ . Hence, we only get the trivial solution  $X(x) \equiv 0$ .

$\lambda > 0$  With  $\lambda = p^2$  the solutions are given by  $X(x) = Ae^{px} + Be^{-px}$ . The boundary conditions  $X(0) = 0$  and  $X'(1) = -3X(1)$  gives us the system

$$\begin{cases} X(0) = A + B = 0 \\ X'(1) + 3X(1) = A(pe^p + pe^{-p}) + 3A(e^p - e^{-p}) = 0, \end{cases}$$

i.e.  $B = -A$  and  $A = 0$  or  $e^p(p+3) + e^{-p}(p-3) = 0$ , but this expression is never 0 for  $p \neq 0$  (show this!) and hence we must have  $A = -B = 0$ , and also in this case we get only the trivial solution  $X \equiv 0$ .

$\lambda < 0$  With  $\lambda = -p^2$  we get the solution  $X(x) = A \cos px + B \sin px$ , and the boundary conditions are  $X(0) = A = 0$ , and  $X'(1) = -3X(1)$  which gives  $pB \cos px = -3B \sin px \Rightarrow$

$$B(p \cos px + 3 \sin px) = 0,$$

hence either  $B = 0$ , (and we get the trivial solution), or

$$(p \cos px + 3 \sin px) = 0,$$

i.e.  $p$  must satisfy the equation  $\tan p = -\frac{p}{3}$ .

Thus, we see that we only have non-trivial solutions when  $\lambda$  is an eigenvalue  $\lambda = \lambda_n = -p_n^2$ ,  $n = 1, 2, \dots$ , where  $p_n$  is a solution of  $\tan p = -\frac{p}{3}$  (see Fig. 5.2.1), and then we have the corresponding eigenfunctions

$$X_n(x) = a_n \sin p_n x.$$

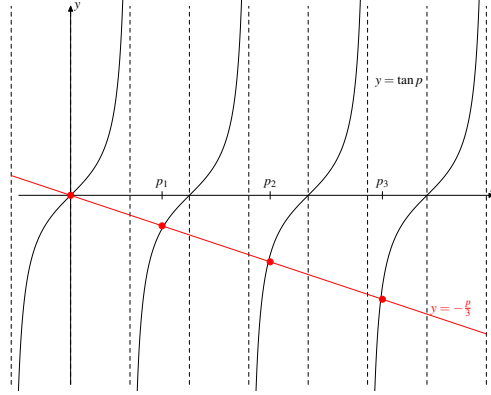
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**Example 5.6.** Solve

$$\begin{cases} x^2 X''(x) + 2x X'(x) + \lambda X = 0, \\ X(1) = 0, X(e) = 0. \end{cases}$$

**Solution:** The characteristic equation is

$$\begin{aligned} r(r-1) + 2r + \lambda &= 0 \\ &\Leftrightarrow \\ r^2 + r + \lambda &= 0 \end{aligned}$$

FIGURE 5.2.1. Solutions to  $\tan p = -\frac{p}{3}$ 

which has the solutions

$$r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}},$$

hence the cases we must investigate are  $\lambda < \frac{1}{4}$ ,  $\lambda = \frac{1}{4}$  and  $\lambda > \frac{1}{4}$  (cf. Example 5.3).

$\lambda < \frac{1}{4}$  With  $r_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$  (different real) we get the solutions  $X(x) = Ax^{r_1} + Bx^{r_2}$  and the boundary conditions gives

$$\begin{cases} X(1) = 0, \\ X(e) = 0, \end{cases} \Leftrightarrow \begin{cases} A + B = 0, \\ Ae^{r_1} + Be^{r_2} = 0, \end{cases} \Leftrightarrow \begin{cases} A = -B, \\ A(e^{r_1} - e^{r_2}) = 0, \end{cases}$$

i.e. since  $e^{r_1} \neq e^{r_2}$  we must have  $A = 0$  and we only get the trivial solution  $X \equiv 0$ .

$\lambda = \frac{1}{4}$  Now we get a double root  $r = -\frac{1}{2}$  and the solutions are  $X(x) = Ax^{-\frac{1}{2}} + Bx^{-\frac{1}{2}} \ln x$ . The boundary conditions give  $X(1) = A = 0$  and  $X(e) = Be^{-\frac{1}{2}} = 0$ , i.e.  $A = B = 0$  and we only get the trivial solution  $X \equiv 0$ .

$\lambda > \frac{1}{4}$  The two complex roots  $r = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}$  give the solutions

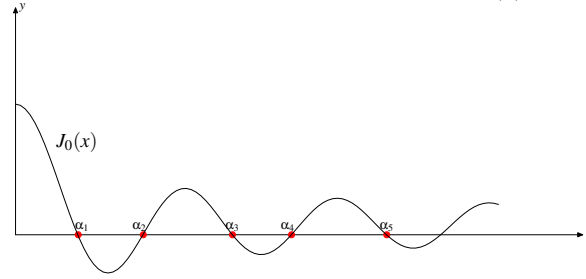
$$X(x) = \frac{A}{\sqrt{x}} \sin\left(\sqrt{\lambda - \frac{1}{4}} \ln x\right) + \frac{B}{\sqrt{x}} \cos\left(\sqrt{\lambda - \frac{1}{4}} \ln x\right),$$

and we get  $X(1) = B = 0$ , and  $X(e) = \frac{A}{\sqrt{e}} \sin\left(\sqrt{\lambda - \frac{1}{4}}\right) = 0$  hence  $\lambda$  must satisfy

$$\sqrt{\lambda - \frac{1}{4}} = n\pi,$$

for some positive integer  $n$ . We get the *eigenvalues*

$$\lambda_n = \frac{1}{4} + (n\pi)^2, \quad n \in \mathbb{Z}^+,$$

FIGURE 5.2.2. The Bessel function  $J_0(x)$ 

and the corresponding *eigenfunctions*

$$X_n(x) = \frac{A_n}{\sqrt{x}} \sin(n\pi \ln x).$$

◇

**Example 5.7.** (The Bessel equation)

An important ordinary differential equation in mathematical physics is the Bessel equation ([Wilhelm Bessel](#)) of order  $m$ :

$$r^2 w'' + r w' + (r^2 - m^2) w = 0.$$

The solutions (there are two linearly independent) to this equation are called Bessel functions of order  $m$ . (For more information see e.g. [Besselfunktionen at engineering fundamentals](#)). Here we will only consider a special case.

Solve the following problem involving the Bessel equation of order 0:

$$\begin{cases} \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + k^2 w = 0, \\ w(R) = 0, w'(r) < \infty. \end{cases}$$

**Solution:** A general solution is given by

$$w(r) = C_1 J_0(kr) + C_2 Y_0(kr),$$

where  $J_0$  and  $Y_0$  are the Bessel functions of the first and second kind of order 0. It is known that  $Y_0'$  is not bounded and if we impose the condition that  $w'(r)$  must be bounded we get  $C_2 = 0$ . The boundary condition implies that

$$w(R) = C_1 J_0(kR) = 0,$$

and if we want a non-trivial solution ( $C_1 \neq 0$ ) then  $k$  and  $R$  must satisfy  $J_0(kR) = 0$ . It is well-known that  $J_0$  has infinitely many zeros  $\alpha_n$  ( $\alpha_1 = 2.4047 \dots$ ,  $\alpha_2 = 5.5201 \dots$ ,  $\alpha_3 = 8.6537 \dots$ ,  $\dots$  etc., see [Fig. 5.2.2](#)). Hence we only get non-trivial solutions for the *eigenvalues*

$$k_n = \frac{\alpha_n}{R}, n \in \mathbb{Z}^+,$$

with the corresponding solutions are the *eigenfunctions*

$$w_n(r) = J_0\left(\frac{\alpha_n}{R} r\right), n \in \mathbb{Z}^+.$$

### 5.3. Inner Product and Norm

To construct an orthonormal basis in a *vector space* we must be able to measure lengths and angles. Hence we must introduce an *inner product* (a *scalar product*). With the help of an inner product we can easily determine which elements are orthogonal to each other. There are two examples of vector spaces and inner products we will consider here. The plane  $\mathbb{R}^2$  together with the usual scalar product, and a vector space consisting of functions on an interval together with an inner product defined by an integral.

#### Vectors in $\mathbb{R}^2$

If we have two vectors  $\vec{x} = (x_1, x_2)$  and  $\vec{y} = (y_1, y_2)$ , the *inner product* of  $\vec{x}$  and  $\vec{y}$  is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2.$$

The *norm* of  $\vec{x}$ ,  $|\vec{x}|$ , is defined by

$$|\vec{x}|^2 = \vec{x} \cdot \vec{x} = x_1^2 + x_2^2,$$

and the distance between  $\vec{x}$  and  $\vec{y}$ ,  $|\vec{x} - \vec{y}|$ , is given by

$$|\vec{x} - \vec{y}|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

The angle  $\theta$  between  $\vec{x}$  and  $\vec{y}$  can now be computed using the relation

$$\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta,$$

and we say that two vectors are *orthogonal* (*perpendicular* to each other),  $\vec{x} \perp \vec{y}$ , if  $\theta = \frac{\pi}{2}$ , i.e. if

$$\vec{x} \cdot \vec{y} = 0.$$

#### A Function Space

We now consider the vector space consisting of functions  $f(x)$  defined on the interval  $[0, l]$  (for some  $l > 0$ ) together with a positive *weight-function*  $r(x)$ . The generalizations of the concepts above are

$$\begin{aligned} \langle f, g \rangle &= \int_0^l f(x)g(x)r(x)dx, & (\text{inner product}) \\ \|f\|^2 &= \int_0^l |f(x)|^2 r(x)dx, & (\text{norm}) \\ \|f - g\|^2 &= \int_0^l |f(x) - g(x)|^2 r(x)dx, & (\text{distance}) \\ \langle f, g \rangle &= \|f\| \|g\| \cos \theta, & (\text{angle}) \\ f \perp g &\Leftrightarrow \langle f, g \rangle = 0 & (\text{orthogonality}) \\ &\Leftrightarrow \int_0^l f(x)g(x)r(x)dx = 0. \end{aligned}$$

### 5.4. Sturm-Liouville Problems

A general *Sturm-Liouville problem* can be written as

$$\begin{aligned} (\mathcal{P}(x)y')' + (-q(x) + \lambda r(x))y &= 0, & 0 < x < l, \\ c_1 y(0) + c_2 y'(0) &= 0, \\ c_3 y(l) + c_4 y'(l) &= 0. \end{aligned}$$

Here  $r(x)$ ,  $q(x)$  and  $\mathcal{P}(x)$  are given functions,  $c_1, \dots, c_4$  given constants and  $\lambda$  a constant which can only take certain values, the eigenvalues corresponding to the problem.  $r(x)$  is usually called a *weight function*. It is also customary to assume that  $r(x) > 0$ .

If  $\mathcal{P}(x) > 0$  and  $c_1, \dots, c_4 \neq 0$  we say that the problem is *regular*, and if  $\mathcal{P}$  or  $r$  is 0 in some endpoint we say that it is *singular* (note that there are other examples of both regular and singular SL problem, e.g. the following problem is regular).

**Example 5.8.**  $r(x) = 1$ ,  $\mathcal{P}(x) = 1$ ,  $q(x) = 0$ ,  $c_1 = c_3 = 1$ ,  $c_4 = c_2 = 0$ .

$$\begin{cases} y'' + \lambda y = 0, \\ y(0) = 0, \\ y(l) = 0. \end{cases}$$

(Cf. Example 5.4). In this case we have

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3, \dots, \quad (\text{eigen values}) \\ y_n &= \sin\left(\frac{n\pi}{l}x\right), \quad (\text{eigen functions}) \end{aligned}$$

and

$$\begin{aligned} \langle y_n, y_m \rangle &= \int_0^l \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{m\pi}{l}x\right) dx = 0, \text{ if } n \neq m, \\ \|y_n\|^2 &= \int_0^l \left|\sin\left(\frac{n\pi}{l}x\right)\right|^2 dx = \int_0^l \frac{1}{2} \left(1 - \cos\left(\frac{2n\pi}{l}x\right)\right) dx = \frac{l}{2}. \end{aligned}$$

If  $f$  is a function on the interval  $[0, l]$  we can define the *Fourier series* of  $f$ ,  $S(x)$ , by (cf. Section 6.1):

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right), \text{ where} \\ c_n &= \frac{1}{\|y_n\|^2} \langle f, y_n \rangle = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx. \end{aligned}$$

REMARK 7. Examples 5.5-5.7 is also a Sturm-Liouville problem.

For a regular Sturm-Liouville problem we have:

- (i) The eigenvalues are real and to every eigenvalue the corresponding eigenfunction is unique up to a constant multiple.
- (ii) The eigenvalues form an infinite sequence  $\lambda_1, \lambda_2, \dots$  and they can be ordered as

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots,$$

with

$$\lim_{n \rightarrow \infty} \lambda_n = \infty.$$

- (iii) If  $y_1$  and  $y_2$  are two eigenfunctions corresponding to two different eigenvalues,  $\lambda_{i_1} \neq \lambda_{i_2}$ , they are orthogonal with the respect to the inner product defined by  $r(x)$ , i.e.

$$\langle y_1, y_2 \rangle = \int_0^l y_1(x)y_2(x)r(x)dx = 0.$$

### 5.5. Generalized Fourier Series

We will now see how we can generalize the concept of Fourier series from the usual trigonometric basis functions to an orthonormal basis consisting of eigenfunctions to a Sturm-Liouville problem.

Assume that we have an infinite linear combination

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

where  $y_n \perp y_m$  for  $n \neq m$ . Then

$$\begin{aligned} \langle f, y_m \rangle &= \left\langle \sum_{n=1}^{\infty} c_n y_n, y_m \right\rangle = \sum_{n=1}^{\infty} c_n \langle y_n, y_m \rangle \\ &= c_m \langle y_m, y_m \rangle = c_m \|y_m\|^2. \end{aligned}$$

Let  $f$  be an arbitrary function on  $[0, l]$ . Then we define the *generalized Fourier series* for  $f$  as

$$\mathcal{S}(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

where

$$c_n = \frac{1}{\|y_n\|^2} \langle f, y_n \rangle$$

are the *generalized Fourier coefficients*.

Let  $y_1, y_2, \dots$  be a set of orthogonal eigenfunctions of a regular Sturm-Liouville problem, and let  $f$  be a piece-wise smooth function in  $[0, l]$ . Then, for each  $x$  in  $[0, l]$  we have that

- (a)  $\mathcal{S}(x) = f(x)$  if  $f$  is continuous at  $x$ , and
- (b)  $\mathcal{S}(x) = \frac{1}{2} (f(x+) + f(x-))$  if  $f$  has a discontinuity point at  $x$ .

### 5.6. Some Applications

**Example 5.9.** Consider a rod of length  $l$ , with constant density  $\rho$ , specific heat  $c_v$  and thermal conductance  $\kappa$ . Let the temperature of the rod at the time  $t$  and the distance  $x$  (from, say, the left end point) be denoted by  $u(x, t)$ . Assume that the temperature at the end points of the rod are given by

$$u(0, t) = u(l, t) = 0, \quad t > 0,$$

and that the temperature distribution in the rod at the initial time  $t = 0$  is given by

$$u(x, 0) = f(x), \quad 0 \leq x \leq l.$$

Determine  $u(x, t)$  for  $0 \leq x \leq l$ , and  $t \geq 0$ .

**Solution:** We have seen (cf. [Chapter 1](#)) that the mathematical formulation of this problem is

$$(*) \quad \begin{cases} u_t'(x, t) - k u_{xx}''(x, t) = 0, & 0 \leq x \leq l, t \geq 0, k = \frac{\kappa}{c_v \rho}, \\ u(0, t) = u(l, t) = 0, & t > 0, \\ u(x, 0) = f(x), & 0 \leq x \leq l. \end{cases}$$

We begin by performing the following natural scaling of the problem (cf. [Chapter 1](#)):

$$(5.6.1) \quad \bar{t} = \frac{k}{l^2} t, \quad \bar{x} = \frac{x}{l}.$$



Then we arrive at the following standard problem to solve:

$$\begin{aligned} (1) & \quad \begin{cases} \tilde{u}'_{\bar{t}}(\bar{x}, \bar{t}) - \tilde{u}''_{\bar{x}\bar{x}}(\bar{x}, \bar{t}) = 0, & 0 \leq \bar{x} \leq 1, \bar{t} \geq 0, \\ \tilde{u}(0, \bar{t}) = \tilde{u}(1, \bar{t}) = 0, & \bar{t} > 0, \\ \tilde{u}(\bar{x}, 0) = \tilde{f}(\bar{x}), & 0 \leq \bar{x} \leq 1, \end{cases} \\ (2) & \\ (3) & \end{aligned}$$

where  $\tilde{f}(\bar{x}) = f(\bar{x}l)$ . We can now use Fourier's method to solve this problem (cf. section 4.8).

**Step 1:** Try to find solutions of the type

$$\tilde{u}(\bar{x}, \bar{t}) = X(\bar{x})T(\bar{t}).$$

If we insert this expression in (1) above we get

$$\frac{T'(\bar{t})}{T(\bar{t})} = \frac{X''(\bar{x})}{X(\bar{x})} = -\lambda,$$

i.e. the two equations

$$\begin{aligned} (A) & \quad X''(\bar{x}) + \lambda X(\bar{x}) = 0, \text{ and} \\ (B) & \quad T'(\bar{t}) + \lambda T(\bar{t}) = 0. \end{aligned}$$

The function  $u(x, t) = X(x)T(t)$  must also satisfy the boundary conditions (2):

$$X(0)T(\bar{t}) = X(1)T(\bar{t}) = 0, \bar{t} \geq 0,$$

and if we want a non-trivial solution ( $T \not\equiv 0$ ) we conclude that

$$X(0) = X(1) = 0.$$

This boundary condition together with (A) gives us the Sturm-Liouville problem

$$(**) \quad \begin{cases} X''(\bar{x}) + \lambda X(\bar{x}) = 0, \\ X(0) = X(1) = 0. \end{cases}$$

**Step 2:** We get three cases depending on the value of  $\lambda$ :  $\lambda < 0$ ,  $\lambda = 0$ , and  $\lambda > 0$ .

$\lambda < 0$  We get only the trivial solution  $X(\bar{x}) \equiv 0$ .

$\lambda = 0$  We get only the trivial solution  $X(\bar{x}) \equiv 0$ .

$\lambda > 0$  Then

$$X(\bar{x}) = A \sin(\sqrt{\lambda}\bar{x}) + B \cos(\sqrt{\lambda}\bar{x}),$$

$$\text{and } X(0) = 0 \Rightarrow B = 0, \text{ and } X(1) = 0 \Rightarrow A \sin(\sqrt{\lambda}) = 0 \Rightarrow A = 0 \text{ or } \sqrt{\lambda} = n\pi, n \in \mathbb{Z}^+.$$

Thus the SL-problem (\*\*) has the following eigenvalues

$$\lambda_n = (n\pi)^2, n \in \mathbb{Z}^+,$$

and the corresponding eigenfunctions

$$X_n(\bar{x}) = \sin(n\pi\bar{x}).$$

Furthermore, for these values of  $\lambda = \lambda_n$ , the equation (B) has the solution

$$T(\bar{t}) = T_n(\bar{t}) = e^{-(n\pi)^2\bar{t}},$$

and we conclude that the general (separable) solution of (1) and (2) can be written as

$$\tilde{u}_n(\bar{x}, \bar{t}) = c_n \sin(n\pi\bar{x}) e^{-(n\pi)^2\bar{t}}.$$

**Step 3:** The superposition principle (cf. section 4.5) tells us that the function

$$\tilde{u}(\bar{x}, \bar{t}) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi\bar{x}) e^{-(n\pi)^2\bar{t}}$$

also satisfy (1) and (2). We will now see that we can also assert that this function satisfy the initial condition (3) by choosing appropriate constants  $\tilde{b}_n$ . It is clear that

$$\tilde{u}(\bar{x}, 0) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi\bar{x}),$$

and if we choose the  $\tilde{b}_n$ 's as the Fourier coefficients of  $\tilde{f}$ , i.e.

$$\tilde{b}_n = 2 \int_0^1 \tilde{f}(\bar{x}) \sin(n\pi\bar{x}) d\bar{x},$$

we actually get

$$\tilde{u}(\bar{x}, 0) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi\bar{x}) = \tilde{f}(\bar{x}).$$

We conclude that the function

$$\tilde{u}(\bar{x}, \bar{t}) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi\bar{x}) e^{-(n\pi)^2 \bar{t}},$$

with  $\tilde{b}_n$  as above satisfy (1), (2) and (3).

**Final step:** By using the scaling from (5.6.1) we see that the solution to the original problem is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) e^{-\left(\frac{n\pi}{l}\right)^2 kt},$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx.$$

◇

**Example 5.10.** Consider a rod between  $x = 1$  and  $x = e$ . Let  $u(x, t)$  denote the temperature of the rod at the point  $x$  and time  $t$ . Assume that the end points are kept at the constant temperature 0, that at the initial time  $t = 0$  the rod has a heat distribution given by

$$u(x, 0) = f(x), \quad 1 < x < e,$$

that no heat is added and that the rod has constant density  $\rho$  and specific heat  $c_v$ . Assume also that the rod has heat conductance  $K$  which varies as  $K(x) = x^2$ . The equation which determines the temperature  $u(x, t)$  is in this case

$$(1) \quad c_v \rho u_t' = \frac{\partial}{\partial x} (x^2 u_x'), \quad 1 < x < e, \quad t > 0.$$

Determine  $u(x, t)$  for  $1 \leq x \leq e$ , and  $t > 0$ .

**Solution:** We apply Fourier's method of separating the variables and assume that we can find a solution of the form  $u(x, t) = X(x)T(t)$ . Inserting this expression in (1) above we get

$$c_v \rho \frac{T'}{T} = \frac{1}{X} \frac{d}{dx} (x^2 X') = -\lambda,$$

where  $\lambda$  is a constant and  $X$  satisfies the boundary condition

$$(2) \quad X(1) = X(e) = 0.$$

Thus  $T$  satisfies the equation

$$(3) \quad T' = -\frac{\lambda}{c_v \rho} T,$$

and  $X$  satisfies

$$\begin{aligned} \frac{d}{dx} (x^2 X') + \lambda X &= 0, 1 < x < e \\ &\Leftrightarrow \\ (4) \quad x^2 X'' + 2x X' + \lambda X &= 0, 1 < x < e. \end{aligned}$$

The equation (4) together with the boundary condition (2) gives us a regular Sturm-Liouville problem on  $[1, e]$ . The characteristic equation is

$$r(r-1) + 2r + \lambda = 0,$$

with the roots

$$r_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

As in Example 5.6 we get three different cases depending on the value of  $\lambda$ :

$\lambda = \frac{1}{4}$  We have a double root  $r = -\frac{1}{2}$ , and the solutions are given by  $X(x) = Ax^{-\frac{1}{2}} + Bx^{-\frac{1}{2}} \ln x$ .

The boundary condition (2) gives  $X(1) = A = 0$  and  $X(e) = Be^{-\frac{1}{2}} = 0$ , i.e. we get only the trivial solution  $X \equiv 0$ .

$\lambda < \frac{1}{4}$  The roots are now real and different,  $r_1 \neq r_2$ , and the solutions are

$$X(x) = Ax^{r_1} + Bx^{r_2}.$$

The boundary conditions gives us

$$\begin{cases} X(1) = A + B = 0 \\ X(e) = Ae^{r_1} + Be^{r_2} = 0 \end{cases} \Rightarrow \begin{cases} A = -B \\ A(e^{r_1} - e^{r_2}) = 0, \end{cases}$$

and since  $r_1 \neq r_2$  we must have  $A = 0$ , and we only get the trivial solution  $X \equiv 0$ .

$\lambda > \frac{1}{4}$  We have two complex roots  $r = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}$ , and the general solution is

$$X(x) = \frac{A}{\sqrt{x}} \sin\left(\sqrt{\lambda - \frac{1}{4}} \ln x\right) + \frac{B}{\sqrt{x}} \cos\left(\sqrt{\lambda - \frac{1}{4}} \ln x\right).$$

The boundary conditions imply that  $X(1) = B = 0$  and  $X(e) = Ae^{-\frac{1}{2}} \sin\left(\sqrt{\lambda - \frac{1}{4}}\right) = 0$ , which us gives that

$$\sqrt{\lambda - \frac{1}{4}} = n\pi, n \in \mathbb{Z}^+.$$

Observe that the case  $n = 0$  is the same as  $\lambda = \frac{1}{4}$ . Hence the eigenvalues of the Sturm-Liouville problem (4) and (2) are

$$\lambda_n = \frac{1}{4} + n^2 \pi^2, n \in \mathbb{Z}^+,$$

and the corresponding eigenfunctions are

$$X_n(x) = \frac{1}{\sqrt{x}} \sin(n\pi \ln x), n \in \mathbb{Z}^+.$$

For every fixed  $n$ , the equation (3) is

$$T_n' = -\frac{\lambda_n}{c_v \rho} T_n,$$

with the solutions

$$T_n(t) = e^{-\frac{\lambda_n}{c\nu\rho}t}, n \in \mathbb{Z}^+.$$

We conclude that the functions

$$u_n(x, t) = T_n(t)X_n(x) = \frac{1}{\sqrt{x}} \sin(n\pi \ln x) e^{-\frac{\lambda_n}{c\nu\rho}t}, n \in \mathbb{Z}^+,$$

are solutions to the original equation, and they also satisfy the boundary conditions. The superposition principle implies that the function

$$u(x, t) = \sum_{n=1}^{\infty} a_n \frac{1}{\sqrt{x}} \sin(n\pi \ln x) e^{-\frac{\lambda_n}{c\nu\rho}t}$$

is also a solution of the equation which satisfy the boundary conditions. Finally, to accommodate the initial values we must choose the constants  $a_n$  so that

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \frac{1}{\sqrt{x}} \sin(n\pi \ln x) = f(x),$$

This holds if we choose the constants  $a_n$  as

$$\begin{aligned} a_n &= \frac{1}{\|X_n\|^2} \int_1^e f(x) X_n(x) dx \\ &= 2 \int_1^e f(x) \frac{1}{\sqrt{x}} \sin(n\pi \ln x) dx. \end{aligned}$$

(Note that  $\|X_n\|^2 = \int_1^e \frac{1}{x} \sin^2(n\pi \ln x) dx = \frac{1}{2}$ .) The wanted temperature distribution is thus given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n \frac{1}{\sqrt{x}} \sin(n\pi \ln x) e^{-\frac{\lambda_n}{c\nu\rho}t},$$

where

$$a_n = 2 \int_1^e \frac{f(x)}{\sqrt{x}} \sin(n\pi \ln x) dx.$$

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**Example 5.11.** Solve the problem:

- (1)  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$
- (2)  $u(0, t) = 0,$
- (3)  $u'_x(1, t) = -3u(1, t),$
- (4)  $u(x, 0) = f(x).$

**Solution:** We use Fourier's method of separation of variables.

**Step 1:** Assume that  $u(x, t) = X(x)T(t)$  and insert this into (1). In the same way as in the previous examples we obtain the equation

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

which gives us the two equations

- (5)  $T'(t) - \lambda T(t) = 0,$
- (6)  $X''(x) - \lambda X(x) = 0.$

**Step 2:** There are three cases of  $\lambda$  we must study.

$\lambda = 0$  The solutions to (5) and (6) are then  $T = \text{constant}$ , and  $X = Ax + B$ , i.e.  $u(x, t) = Ax + B$  for some constants  $A$  and  $B$ . The boundary value (2) gives  $u(0, t) = B = 0$ , and (3) gives  $u'_x(1, t) = A = -3u(1, t) = -3A$ , i.e.  $A = 0$  and we only get the trivial solution  $u(x, t) \equiv 0$ .

$\lambda > 0$  The solutions to (5) are  $T(t) = Ae^{\lambda t}$  and the solutions to (6) are  $X(x) = Be^{\sqrt{\lambda}x} + Ce^{-\sqrt{\lambda}x}$ . The boundary value (2) gives  $u(0, t) = T(t)X(0) = Ae^{\lambda t}(B + C) = 0$ , hence either  $A = 0$  (which implies  $u \equiv 0$ ) or  $B = -C$ . The condition (3) is now equivalent to

$$\begin{aligned} Ae^{\lambda t} \sqrt{\lambda} B (e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}) &= -3AB (e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}), \\ &\Leftrightarrow \\ AB e^{2\sqrt{\lambda}} (3 + \sqrt{\lambda}) &= AB (3 - \sqrt{\lambda}), \end{aligned}$$

which is only satisfied if  $AB = 0$  (show this!), and in this case we only get the trivial solution  $u \equiv 0$ .

$\lambda < 0$  If we set  $\lambda = -p^2$  we get (in the same manner as in Example 5.5) the solutions

$$(*) \quad u_n(x, t) = B_n e^{-p_n^2 t} \sin p_n x, \quad n = 1, 2, 3, \dots,$$

where  $p_n$  are solutions to the equation

$$\tan p = -\frac{p}{3}.$$

**Step 3:** All functions defined by (\*) satisfy (1), (2) and (3). According to the superposition principle the function

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-p_n^2 t} \sin p_n x$$

also satisfies (1), (2) and (3). Furthermore, (6) with the corresponding boundary conditions is a regular Sturm-Liouville problem and the theory of generalized Fourier series implies that  $u(x, t)$  will satisfy (4):

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin p_n x = f(x)$$

if we choose the constants  $B_n$  as

$$(**) \quad B_n = \frac{\langle f(x), \sin p_n x \rangle}{\|\sin p_n x\|^2} = \frac{\int_0^1 f(x) \sin p_n x dx}{\int_0^1 \sin^2 p_n x dx}.$$

Thus, the solution to the problem is

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-p_n^2 t} \sin p_n x,$$

where  $p_n$  are the positive solutions of  $\tan p = -\frac{p}{3}$ ,  $p_1 < p_2 < \dots$  (see Fig. 5.2.1), and  $B_n$  is defined by (\*\*).

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**Example 5.12.** (The wave equation) A vibrating circular membrane with radius  $R$  is described by the following equation together with boundary and initial values:

$$\begin{aligned}
 (1) \quad & u''_{tt} = c^2 (u''_{xx} + u''_{yy}), t > 0, r = \sqrt{x^2 + y^2} \leq R, \\
 (2) \quad & u(R, t) = 0, t > 0, \text{ (fixated boundary)} \\
 (3) \quad & u(r, 0) = f(r), r \leq R, \text{ (initial position)} \\
 (4) \quad & \frac{\partial u}{\partial t}(r, 0) = g(r), r \leq R. \text{ (initial velocity)}
 \end{aligned}$$

Observe that the initial conditions only depend on  $r = \sqrt{x^2 + y^2}$  = the distance from the center of the membrane to the point  $(x, y)$ , and if we introduce polar coordinates

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases}$$

we see that (1) can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

If we also make the assumption that  $u(r, \theta, t)$  is radially symmetric (i.e. that  $u(r, \theta, t)$  is independent of the angle  $\theta$ ) we can write (1) as

$$(1') \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right).$$

To solve the problem we continue as before and use Fourier's method to separate the variables. With the function  $u(r, t) = W(r)G(t)$  inserted into (1') we get the equations

$$\begin{aligned}
 (5) \quad & W'' + \frac{1}{r} W' + k^2 W = 0, 0 \leq r \leq R, \\
 (6) \quad & G'' + (ck)^2 G = 0, t > 0.
 \end{aligned}$$

Furthermore, we get the following boundary values from (2):

$$(7) \quad W(R) = 0,$$

and (5) together with (7) is a regular Sturm-Liouville problem which gives us the eigenfunctions

$$W_n(r) = J_0 \left( \frac{\alpha_n}{R} r \right),$$

where  $\alpha_n = k_n R$  are solutions of  $J_0(kR) = 0$  (see Example 5.7). Observe that if we write (5) in the general form we see that we have the weight function =  $r$ , i.e. the inner product is given by

$$\langle f, g \rangle = \int_0^R f(r)g(r)r dr.$$

By solving (6) for  $k = k_n$  and using the superposition principle we see that

$$(*) \quad u(r, t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{c\alpha_n}{R} t \right) + B_n \sin \left( \frac{c\alpha_n}{R} t \right) \right) J_0 \left( \frac{\alpha_n}{R} r \right)$$

is a solution to (1) and (2). And we can also choose the constants  $A_n$  so that (3) is satisfied, i.e.

$$u(r, 0) = \sum_{n=1}^{\infty} A_n J_0 \left( \frac{\alpha_n}{R} r \right) = f(r),$$

if

$$(**) \quad A_n = \frac{1}{\int_0^R J_0 \left( \frac{\alpha_n}{R} r \right)^2 r dr} \int_0^R f(r) J_0 \left( \frac{\alpha_n}{R} r \right) r dr.$$

In the same way we see that (4) is satisfied if we choose  $B_n$  so that

$$(***) \quad B_n \frac{c\alpha_n}{R} = \frac{1}{\int_0^R J_0\left(\frac{\alpha_n}{R}r\right)^2 r dr} \int_0^R g(r) J_0\left(\frac{\alpha_n}{R}r\right) r dr.$$

Hence, the answer to the problem is given by (\*) where  $A_n$  and  $B_n$  are chosen as in (\*\*) and (\*\*\*) .

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### 5.7. Exercises

5.1. [A] Solve the following S-L problem by determining the eigenvalues and eigenfunctions:

- (a)  $(x^2 u'(x))' + \lambda u(x) = 0, 1 < x < e^L, u(1) = u(e^L) = 0,$   
 (b)  $(x^2 u'(x))' + \lambda u(x) = 0, 1 < x < e^L, u(1) = u'(e) = 0.$

5.2.\* Solve the following S-L problem by determining the eigenvalues and eigenfunctions:

- (a)  $u''(x) + \lambda u(x) = 0, 0 < x < l, u'(0) = u'(l) = 0,$   
 (b)  $u''(x) + \lambda u(x) = 0, 0 < x < l, u'(0) = u(l) = 0.$

5.3. [A] Use Fourier's method to solve the following problem:

$$\begin{aligned} u'_t &= u''_{xx}, & 0 \leq x \leq l, t > 0, \\ u'_x(0, t) &= u'_x(l, t) = 0, & t > 0, \\ u(x, 0) &= f(x), & 0 < x < l. \end{aligned}$$

5.4.\* A rod between  $x = 1$  and  $x = e$  has constant temperature 0 at the endpoints, and at the time  $t = 0$  the heat distribution is given by  $\sqrt{x}, 1 < x < e$ . The rod has a constant density  $\rho$  and constant specific heat  $C$ , but its thermal conductance varies like  $K = x^2, 1 < x < e$ . Formulate an initial and boundary values problem for the temperature of the rod,  $u(x, t)$ . Then use Fourier's method to solve the problem.

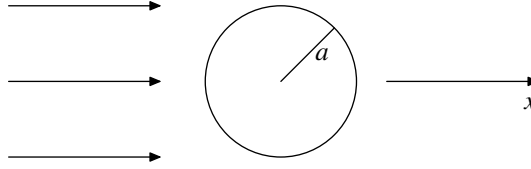
5.5.\*

- (a) Solve the problem

$$\begin{aligned} u'_t &= 4u''_{xx}, & 0 \leq x \leq 1, t > 0, \\ u(0, t) &= 0 & t > 0, \\ u'_x(1, t) &= -cu(1, t), & t > 0, \\ u(x, 0) &= \begin{cases} x, & 0 \leq x < \frac{1}{2}, \\ 1-x, & x \geq \frac{1}{2}. \end{cases} \end{aligned}$$

- (b) Give a physical interpretation of the problem in (a).

- 5.6. [A]** Consider an ideal liquid, flowing orthogonally towards an infinitely long cylinder by the radius  $a$ . Since the problem is uniform in the axial coordinate we can treat the problem in plane polar coordinates.



The speed of the liquid,  $\vec{v}(r, \theta)$  is then given by the equation

$$\vec{v}(r, \theta) = -\text{grad}\psi,$$

where  $\psi$  as a solution of the Laplace equation

$$\Delta\psi = 0.$$

At the surface of the cylinder we have the boundary condition

$$\frac{\partial\psi}{\partial r}\Big|_{r=a} = 0,$$

and as  $r \rightarrow \infty$  we have the following asymptotic boundary condition

$$\lim_{r \rightarrow \infty} \frac{\psi}{x} = \lim_{r \rightarrow \infty} \frac{\psi}{r \cos \theta} = -v_0,$$

where  $v_0$  is a constant.

- a) Show, using separation of variables, that in polar coordinates, the assumption  $\psi(r, \theta) = R(r)\Theta(\theta)$  transforms the Laplace equation to the following two equations

$$\begin{aligned} \Theta''(\theta) + m^2\Theta(\theta) &= 0, \\ R''(r) + \frac{1}{r}R'(r) - \frac{m^2}{r^2}R(r) &= 0, \end{aligned}$$

where  $m$  is an integer.

- b) Use a) to find  $\psi$  and  $\vec{v}$ .