CHAPTER 4

Introduction to Partial Differential Equations

4.1. Some Examples

Example 4.1. (The one-dimensional heat conduction equation)

We consider the heat conduction problem (see Chapter 1) in an (infinitely) thin rod of length l (see Fig. 4.1.1). Let the heat at the point x and time t be given by $u(x,t)$. Assume that the heat distribution in the rod at the time $t = 0$ is given by the function $f(x)$, and that the heat at the endpoints $x = 0$ and $x = l$ are given by the functions $h(t)$ and $g(t)$, respectively (in practice h and g are measured quantities). Then $u(x,t)$ is described by the *heat conduction equation*:

$$
\begin{aligned}
u'_t - ku''_{xx} &= 0, & t > 0, 0 < x < l,\\
u(x,0) &= f(x), & 0 < x < l,\\
u(0,t) &= h(t), & t > 0,\\
u(l,t) &= g(t), & t > 0.
\end{aligned}
$$

FIGURE 4.1.1. One-dimensional heat conduction

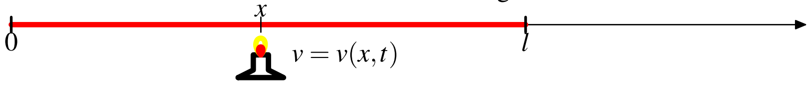
$\diamondsuit$

Example 4.2. (The inhomogeneous one-dimensional heat conduction equation)

Suppose that we have the same system as in the previous example, but that we also add the heat $v(x,t)$ at the point x and time t (see Fig. 4.1.2). In this case $u(x,t)$ is described by the *inhomogeneous heat conduction equation:*

$$
\begin{aligned}
u'_t - ku''_{xx} &= v(x,t), & t > 0, 0 < x < l,\\
u(x,0) &= f(x), & 0 < x < l,\\
u(0,t) &= h(t), & t > 0,\\
u(l,t) &= g(t), & t > 0.
\end{aligned}
$$

FIGURE 4.1.2. One-dimensional inhomogeneous heat conduction



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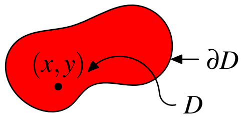
Example 4.3. (The two-dimensional inhomogeneous heat conduction equation)
We now consider heat conduction in a two-dimensional region D. Let the heat at the point $(x,y) \in D$ at the time t be given by $u(x,y,t)$. Assume that the heat distribution at $t = 0$ is described by the function $f(x,y)$, and that the heat at the boundary of D is constant over time and given by $g(x,y)$ (in practice this is obtained by transfer of heat into or out from the system through the boundary). Assume also that the heat $v(x,y,t)$ is added to the point (x,y) at the time t. Then $u(x,y,t)$ is described by the *two-dimensional inhomogeneous heat conduction equation:*

$$\begin{aligned} u'_t - k\left(u''_{xx} + u''_{yy}\right) &= v(x,y,t), \quad (x,y) \in D, t > 0, \\ u(x,y,0) &= f(x,y), \quad (x,y) \in D, \\ u(x,y,t) &= g(x,y), \quad (x,y) \in \partial D, t > 0. \end{aligned}$$



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Example 4.4. (The three-dimensional heat conduction equation)
We now consider heat conduction in a three-dimensional region V. We use the same notation as above, with the addition of a z-coordinate. Then $u(x,y,z,t)$ is described by the *three-dimensional heat conduction equation:*

$$\begin{aligned} u'_t - \operatorname{div}(k \operatorname{grad} u) &= v(x,y,z,t), \quad (x,y,z) \in V, t > 0, \\ u(x,y,z,0) &= f(x,y,z), \quad (x,y,z) \in V, \\ u(x,y,z,t) &= g(x,y,z), \quad (x,y,z) \in \partial V, t > 0. \end{aligned} \tag{4.1.1}$$

◇

REMARK 1. Note that the *gradient* "grad" of the function $u(x,y,z)$ is given by the vector

$$\operatorname{grad} u = \nabla u = \left(u'_x, u'_y, u'_z\right) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) u.$$

If ∇ is written as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right),$$

the *divergence* "div", of a vector field $\vec{F} = (F_x, F_y, F_z)$ is given by

$$\operatorname{div} F = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial z}.$$

Thus, the divergence of the gradient of $u(x,y,z)$ is given by

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot \nabla u = \nabla^2 u = \Delta u = u''_{xx} + u''_{yy} + u''_{zz}.$$

Hence, if $k = k(x,y,z) = k_0$ is constant (4.1.1) can be written as

$$u'_t - k_0\left(u''_{xx} + u''_{yy} + u''_{zz}\right) = v \quad \Leftrightarrow \quad u'_t - k_0 \Delta u = v.$$

REMARK 2. Observe that the equation

$$u'_t - \kappa \Delta u = v$$

in general describes a *diffusion process*. Heat conduction implies a diffusion (transport) of heat, and is one example of such a process. Some other examples of diffusion processes are

- Mixing of one liquid in another (e.g. milk in a cup of tea).
- Diffusion of a gaseous substance in air (e.g. a poisonous gas is released in the air).
- Propagation of elementary particles in a solid material (e.g. neutrinos in a nuclear reactor)

Since the equations are the same, all methods we consider here for solving the heat equation in various cases can also be applied to these alternative diffusion problems. Another PDE which is as important as the diffusion equation is the wave equation, which we will now consider in some examples of.

Example 4.5. (The one-dimensional wave equation)
Consider a vibrating (elastic) string of length l which is fixed at both endpoints. Arrange the string along the x-axis and let $u(x,t)$ describe the position (relative to the equilibrium) of the string at the coordinate x and time t. At the initial time $t = 0$ the position and velocity of the string are given by the functions $f(x)$and $g(x)$ respectively. The vibrations of the string are described by the *one-dimensional wave equation*:

$$\begin{aligned} u''_{tt} - ku''_{xx} &= 0, \quad 0 < x < l, t > 0, \\ u(0,t) = u(l,t) &= 0, \quad t > 0, \\ u(x,0) &= f(x), 0 < x < l, \\ u'_t(x,0) &= g(x), 0 < x < l. \end{aligned}$$

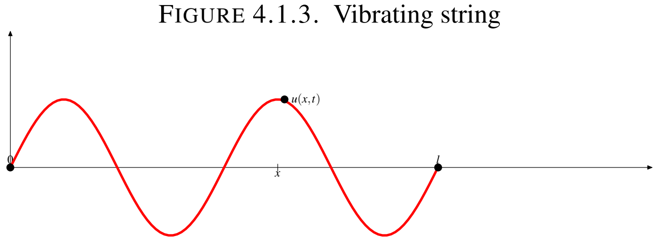

FIGURE 4.1.3. Vibrating string

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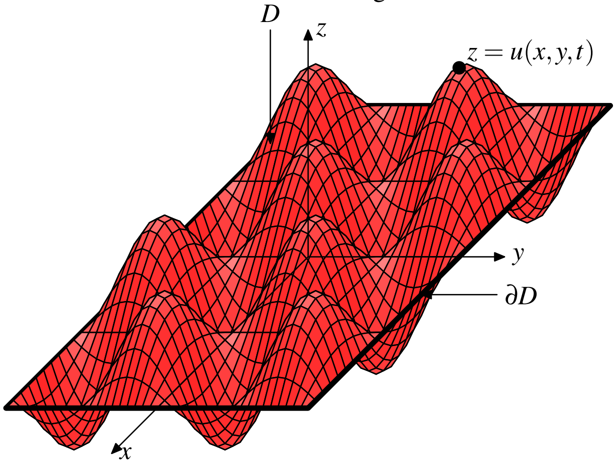
Example 4.6. (The two-dimensional wave equation)
Consider a vibrating membrane which is fixed at the boundary (e.g. a drum skin fastened in a drum). Arrange the membrane so that it covers a domain D in the xy-plane, and let $u(x,y,t)$ describe the position (relative to the equilibrium) of the membrane at the point (x,y) at the time t (see Fig. 4.1.4). At the initial time $t = 0$ the position and velocity of the membrane are given by the functions $f(x,y)$ and $g(x,y)$ respectively. The vibrations of the membrane are then described by the *two-dimensional wave equation:*

$$\begin{aligned} u''_{tt} - k\left(u''_{xx} + u''_{yy}\right) &= 0, & (x,y) \in D, t > 0, \\ u(x,y,t) &= 0, & (x,y) \in \partial D, t > 0, \\ u(x,y,0) &= f(x,y), & (x,y) \in D, \\ u'_t(x,y,0) &= g(x,y), & (x,y) \in D. \end{aligned}$$

FIGURE 4.1.4. Vibrating membrane



$\diamondsuit$

Example 4.7. (The two-dimensional *Laplace equation*)

Assume that we have a two-dimensional domain, as in Example 4.3, and that we want to investigate the heat distribution in the system at *thermal equilibrium,* i.e. after so long time that the heat distribution no longer changes with time. Assume also that we do not add any heat. This implies that we have to set $u'_t = 0$ and $v = 0$ in Example 4.3, which gives us the **Laplace equation**, which we can write in the following three equivalent ways:

$$\begin{aligned} u''_{xx} + u''_{yy} &= 0, \Leftrightarrow \\ \nabla^2 u &= 0, \Leftrightarrow \\ \Delta u &= 0. \end{aligned} \tag{4.1.2}$$

Δ is usually called the *Laplace operator* or simply the *Laplacian*, and is of great importance in both pure and applied mathematics. The solution $u(x,y)$ of (4.1.2) describes the heat in the point (x,y) after thermal equilibrium. This is usually called a *stationary solution* to the heat conduction problem.

$\diamondsuit$

Example 4.8. (The two-dimensional Poisson equation)

The Poisson equation is an inhomogeneous Laplace equation, i.e. at all times t we add the heat $v(x,y,t) = f(x,y)$ (independent of t) to the point (x,y). This equation can be written in the following three equivalent ways:

$$\begin{aligned} u''_{xx} + u''_{yy} &= f \Leftrightarrow \\ \nabla^2 u &= f \Leftrightarrow \\ \Delta u &= f. \end{aligned}$$

Here $u_t' = 0$ and $v(x,y,t) = -\frac{1}{k} f(x,y)$ in Example 4.3, so the the Poisson equation can be interpreted as the inhomogeneous heat conduction equation at thermal equilibrium ($u_t' = 0$), where we at all times t add the heat $f(x,y)$ to the point (x,y).

$\diamondsuit$

Example 4.9. (The three-dimensional Poisson equation)

$$\begin{aligned} u_{xx}'' + u_{yy}'' + u_{zz}'' &= f \Leftrightarrow \\ \nabla^2 u &= f \Leftrightarrow \\ \Delta u &= f. \end{aligned}$$

In this case we have $u_t' = 0$ and $v = -\frac{1}{k_0} f$ in Example 4.4, and as in the two-dimensional case above, the three-dimensional Poisson equation can be interpreted as the heat conduction equation at thermal equilibrium and when at all times t we add the heat $-\frac{1}{k_0} f(x,y,z)$ to the point (x,y,z).

$\diamondsuit$

REMARK 3. If the added heat in the examples above have negative sign, the obvious physical interpretation is that we cool down the system.

4.2. A General Partial Differential Equation of the Second Order

A general partial differential equation (PDE) can be written as

$$G\left(x,t,u,u_x',u_t',u_{xx}'',u_{xt}'',u_{tt}''\right) = 0. \tag{4.2.1}$$

The basic questions we now ask ourselves are:

1. Does it *exist* a solution to the PDE?
2. Is the solution *unique*?
3. Is the solution *stable* under small perturbations?
4. Which methods are available to construct and illustrate solutions?

Example 4.10. The problems in Examples 4.1-4.6 have unique solutions, but the problems in Example 4.7-4.9 do not have unique solutions.

REMARK 4. A PDE of the type (4.2.1) usually has an infinite number of solutions and the general solution depends on a number of arbitrary *functions* (to be compared with the fact that solutions to ODE:s usually depend on arbitrary *constants*).

Example 4.11. The equation

$$u_{tx}'' = tx$$

has the solutions

$$u = \frac{1}{4} t^2 x^2 + g(t) + h(x).$$

$\diamondsuit$

Example 4.12. The two-dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0,$$

has, for example, the solutions

$$\begin{aligned} u(x,y) &= x^2 - y^2, \\ u(x,y) &= e^x \cos y, \\ u(x,y) &= \ln\left(x^2+y^2\right). \end{aligned}$$

REMARK 5. A solution $u(x,y)$ to the Laplace equation is called a *harmonic function*. To find harmonic functions one can use the fact that if $f(z) = f(x+iy)$ is an *analytic function* (or synonymously: entire or holomorphic), i.e. if $\frac{d}{dz}f(z)$ exists, then the real part $u(x,y) = \Re f(x+iy)$, and the imaginary part $v(x,y) = \Im f(x+iy)$ of f are both harmonic functions.
In the example above we used $f(z) = z^2$, e^z and $\log z^2$ respectively.

4.3. Linearity and Non-linearity

A partial differential equation can be written as

$$(*) \qquad Lu = f,$$

where L is a *so called differential operator*.

Example 4.13. Let $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$. Then (*) becomes

$$u'_t - ku''_{xx} = f,$$

which is a one-dimensional heat conduction equation (cf. Example 4.2).

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Example 4.14. Consider the differential operator

$$L(u) = u\frac{\partial u}{\partial t} + 2txu.$$

Then the equation (*) becomes

$$u\frac{\partial u}{\partial t} + 2txu = f(x,t).$$

DEFINITION 4.1. We say that the PDE (*) is *linear* if the operator L has the properties

$$(1) \qquad L(u+v) = Lu + Lv,$$

$$(2) \qquad L(cu) = cLu.$$

If these conditions are not both satisfied we say that (*) is *non-linear*.

Example 4.15. The heat conduction equation in Example 4.13 is linear.

Proof: We must see verify that $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$ satisfies (1) and (2) above.

$$(1) \qquad L(u+v) = \frac{\partial(u+v)}{\partial t} - k\frac{\partial^2(u+v)}{\partial x^2} = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = Lu + Lv.$$

$$(2)\qquad L(cu) = \frac{\partial(cu)}{\partial t} - k\frac{\partial^2(cu)}{\partial x^2} = c\frac{\partial u}{\partial t} - kc\frac{\partial^2 u}{\partial x^2} = c\left(\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}\right) = cLu.$$

Hence, since L satisfies both (1) and (2), the equation

$$Lu = f$$

is linear. $\square$

Example 4.16. The PDE in Example 4.14 is non-linear.

Proof: We start by verifying property (1):

$$\begin{aligned} L(u+v) &= (u+v)\,(u+v)'_t + 2tx\,(u+v) \\ &= uu'_t + uv'_t + vu'_t + vv'_t + 2txu + 2txv, \text{ and} \\ Lu + Lv &= uu'_t + 2txu + vv'_t + 2txv. \end{aligned}$$

Since $L\,(u+v) - (Lu+Lv) = uv'_t + vu'_t \neq 0$ the property (1) is not satisfied and hence the equation is non-linear. $\square$

4.4. Classification of PDEs

A general linear second order PDE can be written as

$$(4.4.1)\qquad a(x,t)u''_{tt} + b(x,t)u''_{xt} + c(x,t)u''_{xx} + d(x,t)u'_t + e(x,t)u'_x + q(x,t)u = f(x,y),\ (x,t) \in \mathcal{D}.$$

Set

$$D(x,t) = (b(x,t))^2 - 4a(x,t)c(x,t).$$

We say that the PDE (4.4.1) is

- **Elliptic** if $D(x,t) < 0$ in $\mathcal{D}$,
- **Parabolic** if $D(x,t) = 0$ in $\mathcal{D}$,
- **Hyperbolic** if $D(x,t) > 0$ in $\mathcal{D}$.

Example 4.17. Consider the two-dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0.$$

Here $D(x,y) = 0^2 - 4\cdot 1\cdot 1 = -4 < 0$, and hence the equation is elliptic.

$\diamondsuit$

Example 4.18. Consider the heat conduction equation

$$u'_t - u''_{xx} = 0.$$

Here $D(x,y) = 0^2 - 4\cdot 0\cdot(-1) = 0$, and hence the equation is parabolic.

$\diamondsuit$

Example 4.19. Consider the one-dimensional wave equation

$$u''_{tt} - u''_{xx} = 0.$$

Here $D(x,y) = 0^2 - 4\cdot 1\cdot(-1) = 4 > 0$, and hence the equation is hyperbolic.

4.5. The Superposition Principle

Consider a linear and homogeneous (i.e. the right hand side is 0) PDE:

$$Lu = 0. \tag{*}$$

Suppose that $u_1, u_2, \ldots$ are solutions of (*) and that u is a finite linear combination of these:

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n.$$

Then u is also a solution to (*) since

$$Lu = L(c_1 u_1 + \cdots + c_n u_n) = c_1 L u_1 + \cdots + c_n L u_n = 0 + \cdots + 0 = 0.$$

This is called the *superposition principle* and is true also for infinite sums:

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n + \cdots,$$

provided that certain convergence properties hold[1].

The continuous superposition principle:

Assume that $u_\alpha(x,t)$ satisfies $Lu_\alpha = 0$ for all α, $a \leq \alpha \leq b$, and define

$$u(x,t) = \int_a^b c(\alpha) u_\alpha(x,t) d\alpha,$$

where $c(\alpha)$ is an arbitrary (integrable) function. Then

$$Lu = 0.$$

Proof:

$$\begin{aligned} Lu &= L\left(\int_a^b c(\alpha) u_\alpha(x,t) d\alpha\right) \\ &= \int_a^b c(\alpha) L u_\alpha(x,t) d\alpha \\ &= \int_a^b c(\alpha) \cdot 0 d\alpha = 0. \end{aligned}$$

□

Example 4.20. It is easy to verify that for each $-\infty < \alpha < \infty$, the function

$$u_\alpha(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-\alpha)^2}{4kt}\right)$$

satisfies the heat conduction equation

$$u'_t - k u''_{xx} = 0.$$

Hence this equation is also satisfied by the function

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} c(\alpha) \exp\left(-\frac{(x-\alpha)^2}{4kt}\right) d\alpha,$$

for any arbitrary, integrable function $c(\alpha)$.

[1] E.g. if we have uniform convergence in: $s_n(x) = \sum_1^n u_j(x) \to u$, $s'_n(x) = \sum_1^n u'_j(x) \to u'$, etc. for all occurring derivatives.

4.6. Well-Posed Problems

A boundary or initial value problem is said to be *well-posed* if

(a) there *exists* a solution,
(b) the solution is *unique*, and
(c) the solution is *stable*.

A problem that is not well-posed is said to be *ill-posed*.

Example 4.21. Consider the initial-values problem which consists of the equation

$$u''_{tt} + u''_{xx} = 0, \, t > 0, \, -\infty < x < \infty,$$

together with the initial-values

$$u(x,0) = 0, \, u'_t(x,0) = 0, \, -\infty < x < \infty. \tag{4.6.1}$$

The unique solution is given by the function which is constant 0:

$$u(x,t) \equiv 0, \, t \geq 0, \, -\infty < x < \infty.$$

Let us now make a little perturbation of the initial-values (4.6.1):

$$u(x,0) = 0, \, u'_t(x,0) = 10^{-4} \sin 10^4 x. \tag{4.6.2}$$

The solution to this new problem is given by

$$u(x,t) = 10^{-8} \sin\left(10^4 x\right) \sinh\left(10^4 t\right).$$

For large t we know that $\sinh\left(10^4 t\right)$ is approximately $\dfrac{1}{2}\exp\left(10^4 t\right)$. The tiny change in the initial-values ave rise to a change in the solution from the constant 0 to a function which grows exponentially (from the sinh-factor) and oscillates exponentially much (from the sine-factor). A really dramatic change! This implies that the solution is not stable, and hence the problem is ill-posed.

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Example 4.22. Show that the boundary-value problem

$$\begin{cases} u'_t - k u''_{xx} = 0, & 0 < x < l, 0 < t < T, \\ u(x,0) = f(x), & 0 < x < l, \\ u(0,t) = g(t), u(l,t) = h(t), & 0 < t < T, \end{cases}$$

where $f \in \mathcal{C}[0,l]$ and $g,h \in \mathcal{C}[0,T]$, has a unique solution, $u(x,t)$, in the rectangle

$$\mathcal{R} : 0 \leq x \leq l, 0 \leq t \leq T.$$

Solution: Later on we will construct a solution to this problem (in Example 5.9)!

But for now, assume that we have *two* different solutions to the problem: $u_1(x,t)$ and $u_2(x,t)$. It is then clear that the function

$$w(x,t) = u_1(x,t) - u_2(x,t)$$

must satisfy the boundary-value problem:

$$\begin{cases} w'_t - k w''_{xx} = 0, & 0 < x < l, 0 < t < T, \\ w(x,0) = 0, & 0 < x < l, \\ w(0,t) = w(l,t) = 0, & 0 < t < T. \end{cases}$$

We now form the "energy integral"

$$E(t) = \int_0^l w^2(x,t)dx.$$

Observe that $E(t) \geq 0$, $E(0) = 0$, and

$$\begin{aligned} E'(t) &= \int_0^l 2ww_t' dx = 2k \int_0^l ww_{xx}'' dx \\ &= \left[2kww_x'\right]_0^l - 2k \int_0^l \left(w_x'\right)^2 dx \\ &= -2k \int_0^l \left(w_x'\right)^2 dx \leq 0. \end{aligned}$$

Hence, the function E is decreasing from $E(0) = 0$, and since $E \geq 0$ we must have $E(t) \equiv 0$. This implies that also $w(x,t) \equiv 0$, i.e. $u_1(x,t) = u_2(x,t)$ for all x,t. Since we assumed that the solutions u_1 and u_2 were different we have arrived at a *contradiction*! Hence the problem must have a unique solution! $\diamondsuit$

4.7. Some Remarks On Fourier Series

Consider a function $f(x)$, $-l < x < l$. The *Fourier coefficients* of f are defined as

$$\begin{aligned} a_0 &= \frac{1}{2l} \int_{-l}^l f(x)dx, \\ a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \, n = 1,2,\ldots, \\ b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \, n = 1,2,\ldots, \end{aligned}$$

and the *Fourier series* of f is defined by

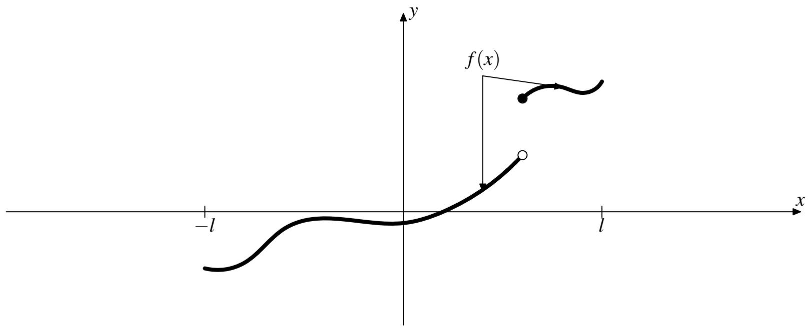
$$\mathcal{S}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right).$$

For a more detailed discussion of Fourier series see Section 6.1. See also Fig. 4.7.

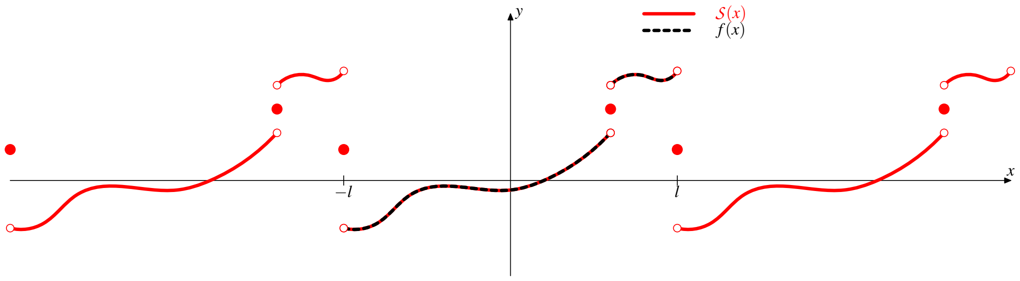
Assume that $f(x)$ is infinitely many times differentiable in the interval $-l < x < l$, except for a number of discontinuity points. Then we have:

(a) $\mathcal{S}(x) = \mathcal{S}(x + 2l)$, for all x.
(b) $\mathcal{S}(x) = f(x)$ at the points where f is continuous,
(c) $\mathcal{S}(x) = \frac{1}{2}\left[f(x+) + f(x-)\right]$ at points of discontinuity[2].

[2] Here $f(+x) = \lim_{y \to x} f(y)$, where we keep $y > x$ as we take the limit, and we define $f(x-)$ similarly.



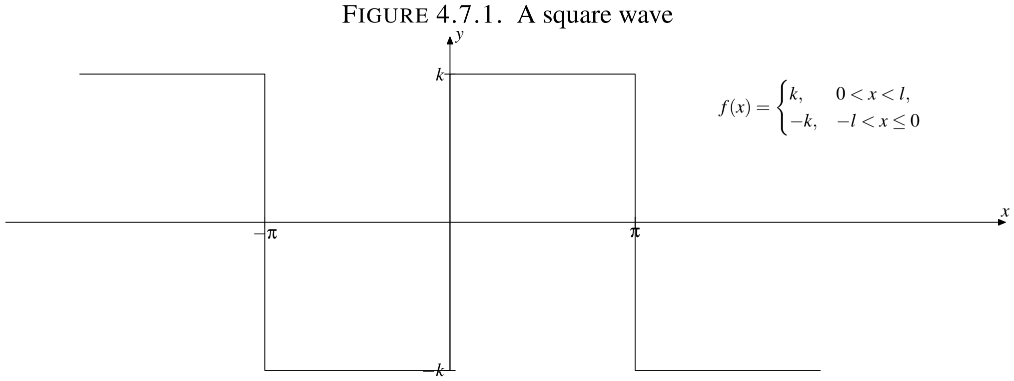
(a) A Discontinuous Function



(b) And its Fourier Series

When making a graph of a discontinuous function it is customary to indicate the value which is attained by the function with a filled circle and the value which is not attained by an unfilled circle.

FIGURE 4.7.1. A square wave



Example 4.23. Consider the function $f(x)$ from Fig. 4.7.1:

$$f(x)=\begin{cases} k, & 0<x<l, \\ -k, & -l<x\le 0. \end{cases}$$

Note that $f(x)$ is odd, i.e. $f(-x)=-f(x)$. Since $\cos x$ is even the function $f(x)\cos\left(\frac{n\pi x}{l}\right)$ is odd and we know that the integral of an odd function over an even interval is always 0 (the “negative” area cancels the "positive" area), hence $a_0 = a_n = 0$ for all n. And we have

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{1}{l}\int_{-l}^{0} -k\sin\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{l} k\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2k}{l}\int_{0}^{l}\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2k}{l}\left[-\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \\ &= \frac{2k}{n\pi}(1-\cos n\pi) \\ &= \frac{2k}{n\pi}(1-(-1)^n). \end{aligned}$$

I.e.

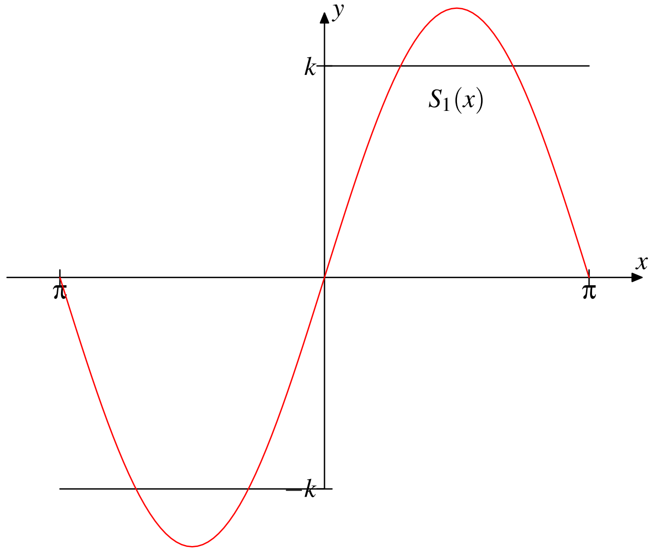
$$b_1 = \frac{4k}{\pi},\, b_2 = 0,\, b_3 = \frac{4k}{3\pi},\, b_4 = 0,\, b_5 = \frac{4k}{5\pi},\ldots,$$

and the Fourier series of f is

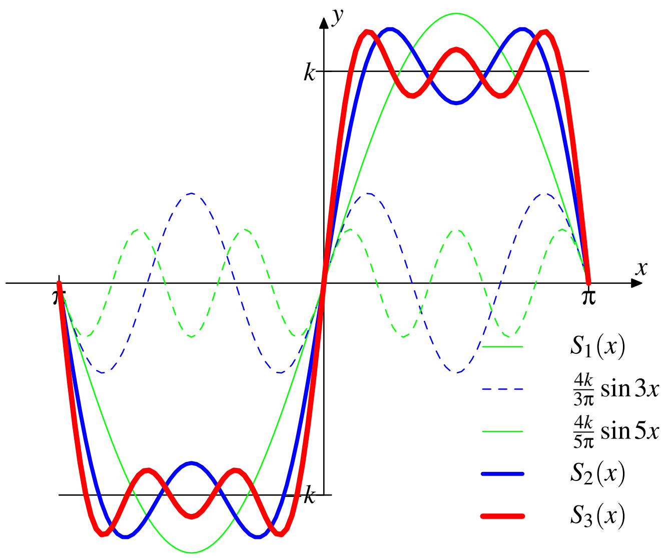
$$\begin{aligned} \mathcal{S}(x) &= \frac{4k}{\pi}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{4k}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\left(\frac{(2m+1)\pi x}{l}\right) \\ &= \frac{4k}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) + \cdots\right). \end{aligned}$$

See Fig. 4.7.2 for an illustration of some of the partial (containing only a finite number of terms) sums for $\mathcal{S}(x)$.

FIGURE 4.7.2. Partial Fourier series



(a) The First Term



(b) More Terms

4.8. Separation of Variables

Separation of variables is a common method to solve certain types of PDEs. Since it originated from an idea of Fourier it is also sometimes called Fourier's method.

Model example: Solve the problem

$$u'_t - ku''_{xx} = 0, \quad 0 < x < l, t > 0, \tag{1}$$

$$u(x,0) = f(x), 0 < x < l, \tag{2}$$

$$u(0,t) = u(l,t) = 0, \quad t > 0. \tag{3}$$

What we mean by separating the variables in (1) is to seek a solution $u(x,t)$ which can be factored as

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ are functions depending only on x and t respectively. Assume now that we can write u in this way. If we differentiate $u = XT$ we get $u'_t(x,t) = X(x)T'(t)$ and $u''_{xx}(x,t) = X''(x)T(t)$, and if we substitute these expressions into (1) we get the equation:

$$X(x)T'(t) - kX''(x)T(t) = 0,$$

which can be rewritten as

$$\frac{T'(t)}{T(t)}\frac{1}{k} = \frac{X''(x)}{X(x)}.$$

Wee see that the left hand side is a function of t only and the right hand side is a function of x only. Hence, the the only possibility is that both sides equals a **constant**:

$$\frac{T'(t)}{T(t)}\frac{1}{k} = \frac{X''(x)}{X(x)} = -\lambda,$$

for some constant λ (which we have to determine later). Instead of the PDE (1) we now have two ODEs:

$$\begin{cases} T'(t) & = -\lambda k T(t), \\ X''(x) & = -\lambda X(x), \end{cases}$$

with the general solutions

$$T(t) = Ce^{-\lambda kt}, \text{and } X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right).$$

The boundary values (3) implies that either $T \equiv 0$ or $X(0) = X(l) = 0$. Since the first alternative only gives us the solution which is constant 0 we see that X must satisfy the boundary conditions $X(0) = X(l) = 0$, i.e.

$$X(0) = B = 0,$$

which tells us that $B = 0$, and we also see that

$$X(l) = A\sin\left(\sqrt{\lambda}l\right) = 0.$$

To once again avoid the trivial solution $W \equiv 0$ (i.e. with $A = 0$) we must have $\sin\left(\sqrt{\lambda}l\right) = 0$, which implies that

$$\sqrt{\lambda}l = n\pi, n \in \mathbb{Z}^+,$$

or equivalently

$$\lambda = \frac{n^2\pi^2}{l^2},$$

for some positive integer n.

We have showed that if a solution to (1) can be factored as $X(x)T(t)$ then it can be written as

$$K\sin\left(\frac{n\pi}{l}x\right)\exp\left(-\frac{n^2\pi^2 kt}{l^2}\right),$$

where n is a positive integer and K a constant. By the superposition principle (sec 4.5) the general solution to (1) satisfying the boundary-values (3) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right),$$

where the Fourier coefficients, $\{b_n\}_{n=1}^{\infty}$, are determined by the initial condition (2):

$$(*) \qquad u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right).$$

Let us for simplicity assume that $l = \pi$ and consider some examples of initial values $f(x)$ in the above model example.

Example 4.24. Let $f(x) = 2\sin x + 4\sin 3x$. Then (*) is satisfied if $b_1 = 2$, $b_2 = 0$, $b_3 = 4$, $b_4 = b_5 = \cdots = 0$. Hence, the solution to the model example is

$$u(x,t) = 2\sin(x)e^{-xt} + 4\sin(3x)e^{-9kt}.$$

$\diamondsuit$

Example 4.25. Let $f(x) = 1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$. Then (*) is satisfied if $b_1 = \frac{4}{\pi}$, $b_2 = 0$, $b_3 = \frac{4}{\pi}\frac{1}{3}$, $b_4 = 0$, $b_5 = \frac{4}{\pi}\frac{1}{5}$, $b_6 = 0$, etc. in this case, the solution to the model example is given by

$$\begin{aligned} u(x,t) &= \frac{4}{\pi}\left(\sin(x)e^{-kt} + \frac{1}{3}\sin(3x)e^{-9kt} + \frac{1}{5}\sin(5x)e^{-25kt} + \cdots\right) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty} \sin\left((2n-1)x\right)e^{-(2n-1)^2 kt}. \end{aligned}$$

$\diamondsuit$

Example 4.26. If we have an arbitrary initial-value function $f(x)$, $0 \le x \le \pi$, the solution to the model example is given by

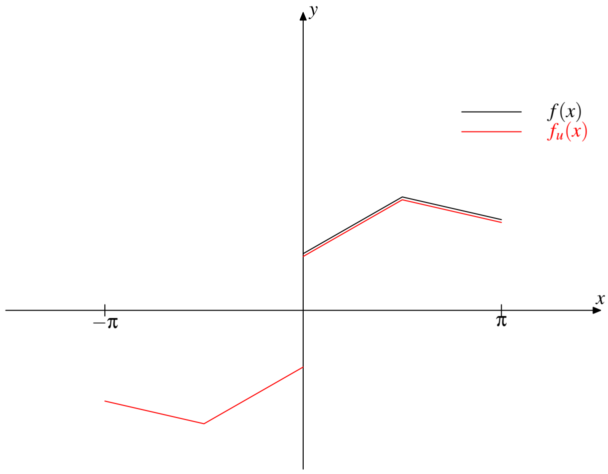
$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) \exp\left(-n^2 kt\right),$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_u(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx.$$

Here $f_u(x)$ is an extension of $f(x)$ to an odd function in the interval $-\pi < x < \pi$, i.e. $f_u(x) = f(x)$ if $x > 0$ and $f_u(x) = -f(x)$ if $x \le 0$ (cf. Fig. 4.8.1).

FIGURE 4.8.1. Construction of an odd extension of a function



4.9. Exercises

4.1. **[A]** Determine, for each of the following differential equations, if it is linear or non-linear:

a) $u'_t(x,t) + x^2 u''_{xx}(x,t) = 0.$

b) $\dfrac{\partial^2 u}{\partial^2 t} + u\dfrac{\partial u}{\partial x} = f(x,t).$

c) $u\Delta u - u'_t = 0.$

d) $\dfrac{\partial^3 u}{\partial^3 t} + \dfrac{\partial^2 u}{\partial^2 t} + \dfrac{\partial u}{\partial t} = u'_x.$

4.2.* Determine, for each of the following partial differential equations, the regions where it is hyperbolic, elliptic or parabolic:

a) $u''_{tt} + xu''_{xx} + 2u'_x = f(x,t),\ (x,t) \in \mathbb{R}^2.$

b) $y^2\left(u''_{xx} + u''_{yy}\right) = 0,\ x \in \mathbb{R},\ y > 0.$

c) $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right),\ t > 0,\ r > 0,$ and $c \in \mathbb{R}$ a constant.

d) $\sin x\left(u''_{tt} + 2u''_{xt}\right) + \cos x u''_{xx} = \tan x,\ t \in \mathbb{R}, |x| \le \pi.$

4.3. **[A]** Let $u(x,t)$, $t > 0$, $x > 0$ denote the temperature in an infinitely long rod with heat conductance coefficient k, and which we heat up by increasing the temperature at the end point such that $u(0,t) = t$. Use the fact that $u_\alpha(x,t) = (4\pi kt)^{-\frac{1}{2}} e^{-\frac{(x-\alpha)^2}{4kt}}$ is a solution of $u'_t - ku''_{xx} = 0$ for each $\alpha \in \mathbb{R}$ together with the superposition principle to determine $u(x,t)$. I.e. solve the problem

$$\begin{aligned} u'_t - ku''_{xx} &= 0,\ x > 0, t > 0, \\ u(0,t) &= t,\ t > 0. \end{aligned}$$

4.4. Determine whether the following problems are well-posed or ill-posed:

a) $u''_{tt} = u''_{xx}$, $u(0,t) = u(\pi,t) = u(x,0) = u(x,\pi)$, $x,t \in [0,\pi]$.

b) $u'_t - ku''_{xx} = 0$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin x$, $x \in [0,\pi]$, $t > 0$.

c) $u'_t - ku''_{xx} = 0$, $u(0,t) = 0$, $u(x,0) = \sin\left(\frac{x}{2}\right)$, $x \in [0,\pi]$, $t > 0$.

c) $u'_t - ku''_{xx} = 0$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin\left(\frac{x}{2}\right)$, $x \in [0,\pi]$, $t > 0$.

4.5. [A] a) Determine the Fourier series for the function $f(x)$ which in the interval $-\pi < x < \pi$ is given by $f(x) = x^2$.

b) use a) to show that $\frac{\pi^2}{12} = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$.

4.6.* Determine the Fourier series of $f(t) = |\sin t|$.

4.7. [A] Consider a rod of length $L = 1$ with heat conduction coefficient $k = 1$. At the beginning the rod has the constant temperature 1. We then (instantaneous) cool down the ends of the rod to the temperature 0, where we then keep it during the continuation of the experiment.

a) Formulate this problem mathematically.

b) find an expression for the temperature of the rod in the point x at the time t.
(Hint: for the Fourier series expansion of the constant 1 use an *odd* periodic extension in the interval.)

4.8.* Solve the following problem by separation of variables:

$$\begin{aligned} u'_t &= u''_{xx},\ 0 < x < 3,\ t > 0, \\ u(x,0) &= \sin(\pi x) - 2\sin\left(\frac{4\pi}{3}x\right),\ 0 < x < 3, \\ u(0,t) &= u(3,t) = 0,\ t > 0. \end{aligned}$$

4.9. [A] Solve the following problem

$$\begin{aligned} u'_t &= u''_{xx},\ 0 < x < \pi,\ t > 0, \\ u(x,0) &= \sin^2 x,\ 0 < x < \pi, \\ u'_x(0,t) &= u'_x(\pi,t) = 0,\ t > 0. \end{aligned}$$

4.10. Solve the following problem

$$\begin{aligned} u''_{tt} &= u''_{xx},\ 0 < x < \pi,\ t > 0, \\ u(x,0) &= \sin x,\ 0 < x < \pi, \\ u'_t(x,0) &= 1,\ 0 < x < \pi, \\ u(0,t) &= u(\pi,t) = 0,\ t > 0. \end{aligned}$$