CHAPTER 6

Introduction to Transform Theory with Applications

6.1. Transforms of Fourier Series Type

Example 6.1. (The classical form)

If $f(t)$ is defined for $t \in [-l, l]$ (or alternatively periodic with period $2l$) we can construct a (classical) Fourier series (Joseph Fourier) for f:

$$\mathcal{F}_{cl}: \quad f(t) \to \quad \{a_0, a_1, b_1, \dots, a_n, b_n, \dots\},$$

where

$$\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(t)dt, \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(t)\cos(n\Omega t)dt, \; n = 1, 2, \dots, \text{ and} \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(t)\sin(n\Omega t)dt, \; n = 1, 2, \dots, \end{aligned}$$

are the Fourier coefficients (amplitudes). Here we have defined $\Omega = \frac{2\pi}{l}$. The "signal" $f(t)$ can be reconstructed (in points of continuity) in the following way:

$$\mathcal{F}_{cl}^{-1}: f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\Omega t) + b_n\sin(n\Omega t).$$

DEFINITION. (Generalized form)

In the generalized form we use, for example, eigenfunctions from a Sturm-Liouville problem (chapter 5.4) instead of the sine and cosine functions.

Let $\{y_n(t)\}_{n=1}^{\infty}$ be an orthogonal system (basis functions), i.e.

$$\langle y_n, y_m \rangle = \begin{cases} 0, & n \neq m, \\ \|y_n\|^2, & n = m. \end{cases}$$

We then define

$$\mathcal{F}_d: f(t) \to \{a_n^*\}_{n=1}^{\infty},$$

where

$$a_n^* = \frac{1}{\|y_n\|^2}\langle f, y_n \rangle$$

are the Fourier coefficients. Under rather general assumptions we can reconstruct the signal $f(t)$ (in points of continuity) by

$$\mathcal{F}_d^{-1}: \quad f(t) = \sum_{n=1}^{\infty} a_n^* y_n(t).$$

REMARK 8. The classical form in Example 6.1 is obtained by considering

$$\{y_n(t)\}_{n=1}^{\infty} = \{1, \cos\Omega t, \sin\Omega t, \dots, \cos n\Omega t, \sin n\Omega t, \dots\}.$$

Note that in this case we have

$$\begin{aligned}\langle f, \cos n\Omega t\rangle &= \int_{-l}^{l} f(t)\cos n\Omega t\,dt, \text{ and}\\ \|\cos n\Omega t\|^2 &= \int_{-l}^{l} \cos^2 n\Omega t\,dt = \int_{-l}^{l} \frac{1+\cos 2\Omega t}{2}dt = l.\end{aligned}$$

Observe also that the integrals can be taken over any period of f, e.g. $[0, 2l]$.

Example 6.2. (Classical complex form)

$$\mathcal{F}_c : f(t) \to \{c_n\}_{n=-\infty}^{\infty},$$

where

$$\begin{aligned}c_0 &= \frac{1}{2l}\int_{-l}^{l} f(t)dt, \text{ and}\\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(t)e^{-in\Omega t}dt, n = \pm1, \pm2, \dots.\end{aligned}$$

Here we have the reconstruction formula:

$$\mathcal{F}_c^{-1} : \quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t}.$$

REMARK 9. The complex form in Example 6.2 can be deduced from the formulas in Example 6.1 and Euler's formulas:

$$\text{(6.1.1)} \qquad \begin{cases}\sin t &= \dfrac{e^{it}-e^{-it}}{2i},\\ \cos t &= \dfrac{e^{it}+e^{-it}}{2},\end{cases} \quad \text{or equivalently} \quad \begin{cases}e^{it} &= \cos t + i\sin t,\\ e^{-it} &= \cos t - i\sin t.\end{cases}$$

We have

$$\begin{aligned}f(t) &= a_0 + \sum_{n=1}^{\infty} a_n\cos n\Omega t + b_n \sin n\Omega t\\ &= a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{in\Omega t}+e^{-in\Omega t}}{2}\right) + b_n\left(\frac{e^{in\Omega t}-e^{-in\Omega t}}{2i}\right)\\ &= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)e^{in\Omega t} + \left(\frac{a_n}{2}-\frac{b_n}{2i}\right)e^{-in\Omega t}\\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{in\Omega t} + \overline{c_n}e^{-in\Omega t},\end{aligned}$$

where we let $c_0 = a_0$ and $c_n = \left(\frac{a_n}{2}+\frac{b_n}{2i}\right)$. (Observe that a_n and b_n are real numbers). If we additionally define

$$c_{-n} = \overline{c_n},$$

we get

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t}.$$

Moreover:

$$
\begin{aligned}
n>0: &\quad c_n = \frac{a_n}{2} - i\frac{b_n}{2} = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-in\Omega t}dt,\\
n=0: &\quad c_0 = a_0,\\
n<0: &\quad c_n = \overline{c_{-n}} = \frac{a_{-n}}{2} - i\frac{b_{-n}}{2} = \frac{1}{2l}\int_{-l}^{l} f(t)\cos(-n\Omega t)dt - \frac{i}{2l}\int_{-l}^{l} f(t)\sin(-n\Omega t)dt\\
&\quad = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-in\Omega t}dt.
\end{aligned}
$$

6.2. The Laplace Transform

If $f(t)$ is defined for $t \geq 0$ the (unilateral) Laplace transform (Pierre-Simon Laplace) $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$ are defined by:

$$
\begin{aligned}
\mathcal{L}: \quad f(t) \mapsto F(s) &\quad = \mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)dt,\\
\mathcal{L}^{-1}: \quad F(s) \mapsto f(t) &\quad = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds.
\end{aligned}
$$

Note that if $f(t)e^{-\sigma_0 t} \to 0$ as $t \to \infty$ then the first integral converges for all complex numbers s with real part greater than σ_0, and in the second integral we then demand that $a > \sigma_0$.

REMARK 10. In applications the inverse transforms are usually computed by using a table (see e.g. Appendix A-1, p. 90). When computing the inverse transform it is sometimes also useful to remember how to compute partial fraction decompositions (see e.g. Appendix A-6, p. 99)

It is obvious that the Laplace transform is *linear*, i.e.

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Apart from computing the Laplace transform of a function by using the integral in the definition above one can also use the general properties stated below, which also illustrate some important properties of the Laplace transform.

Differentiation

$$
\begin{aligned}
\mathcal{L}\{f'(t)\}(s) &= s\mathcal{L}\{f(t)\}(s) - f(0),\\
\mathcal{L}\{f''(t)\}(s) &= s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0),\\
&\vdots\\
\mathcal{L}\left\{f^{(n)}(t)\right\}(s) &= s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).
\end{aligned}
$$

Convolution

The Convolution product of two functions f and g, $f\star g$ over a finite interval $[0,t]$ is defined as

$$(f\star g)(t) = \int_0^t f(u)g(t-u)du.$$

For the Laplace transform we then have

$$\boxed{\mathcal{L}\{f\star g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.}$$

In fact

$$\begin{aligned}
\mathcal{L}\{f\star g\} &= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st}dt \\
&= \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu \\
&= \int_0^\infty f(u)e^{-su}\left(\int_u^\infty g(t-u)e^{-s(t-u)}dt\right)du \\
\{x=t-u\} &= \int_0^\infty f(u)e^{-su}\left(\int_0^\infty g(x)e^{-sx}dx\right)du \\
&= \mathcal{L}\{f\}\mathcal{L}\{g\}.
\end{aligned}$$

Observe that in the second equality we used the following identity: $\int_0^\infty \int_0^t dudt = \int_0^\infty \int_u^\infty dtdu$, which follows from the fact that both sides represent an area integral in the (u,t)-plane over the octant between the positive t-axis and the line $t=u$.

Damping

By damping a "signal" $f(t)$ exponentially, i.e. multiply $f(t)$ with e^{-at} one obtains a translation of the Laplace transform of f as

$$\begin{aligned}
\mathcal{L}\left\{e^{-at}f(t)\right\}(s) &= \int_0^\infty e^{-at}f(t)e^{-st}dt \\
&= \int_0^\infty f(t)e^{-(s+a)t}dt = \mathcal{L}\{f\}(s+a).
\end{aligned}$$

I.e. we have the following formula:

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\}(s) = \mathcal{L}\{f\}(s+a).} \tag{6.2.1}$$

Time delay

Heaviside's function is defined by

$$\theta(t) = \begin{cases} 0, & t<0, \\ 1, & t\geq 0, \end{cases}$$

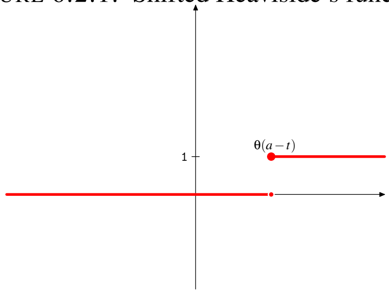
and for $a\in\mathbb{R}$ the function $t\mapsto\theta(t-a)$ is a function which takes the value 0 when $t<a$ and 1 when $t\geq a$ (see Fig. 6.2.1). The meaning of the function $\theta(t-a)$ is to switch on a signal at time $t=a$, and one can also form the function $\theta(t-a)-\theta(t-b)$ which switch on a signal at the time $t=a$ and switch it off at the time $t=b$:

$$f(t)\left(\theta(t-a)-\theta(t-b)\right) = \begin{cases} f(t), & a\leq t\leq b, \\ 0, & \text{else.} \end{cases}$$

Another use of the Heaviside's function is time delay. To translate a function $f(t)$ which is defined for $t\geq 0$ (i.e. delay the signal) one can form the function $t\mapsto f(t-a)\theta(t-a)$, the function which is 0 when $t<a$ and $f(t-a)$ when $t\geq a$. The Laplace transform of this function is given in the following manner by a damping at the transform side

$$\begin{aligned}
\mathcal{L}\{f(t-a)\theta(t-a)\}(s) &= \int_0^\infty f(t-a)\theta(t-a)e^{-st}dt \\
&= \int_a^\infty f(t-a)e^{-st}dt = [u=t-a] \\
&= \int_0^\infty f(u)e^{-s(u+a)}du = e^{-as}\mathcal{L}\{f\}(s),
\end{aligned}$$

FIGURE 6.2.1. Shifted Heaviside's function



i.e. we have the relation

$$\boxed{\mathcal{L}\{f(t-a)\theta(t-a)\}(s) = e^{-as}\mathcal{L}\{f\}(s).} \tag{6.2.2}$$

Example 6.3. We have $\mathcal{L}\{1\}(s) = \frac{1}{s}$, $\mathcal{L}\{t\}(s) = \frac{1}{s^2}, \ldots, \mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$, since

$$\begin{aligned}
\mathcal{L}\{1\}(s) &= \int_0^\infty 1e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s},\\
\mathcal{L}\{t\}(s) &= \int_0^\infty te^{-st}dt = \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty 1e^{-st}dt\\
&= 0 + \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}, \text{ etc.}
\end{aligned}$$

Observe that when we calculate the integral from 0 to ∞ of $t^n e^{-st}$ each integration by parts will give us an s in the denominator and a factor in the numerator, and since $t^k e^{-st}$ vanishes at both limits of the integral (when $k > 0$) all terms will vanish except the last, $\frac{n!}{s^n}\int_0^\infty e^{-st}dt$.

By using this example and the dampening formula (6.2.1) we can easily compute for example

$$\mathcal{L}\left\{e^{-at}\right\}(s) = \frac{1}{s-a},\ \mathcal{L}\left\{te^{-at}\right\}(s) = \frac{1}{(s-a)^2}, \text{ etc.}$$

♢

Example 6.4. Let $f(t) = e^{iat}$, where a is a constant and $t \geq 0$. The Laplace transform of f is then given by:

$$\begin{aligned}
\mathcal{L}\left\{e^{iat}\right\}(s) &= \int_0^\infty e^{iat}e^{-st}dt = \int_0^\infty e^{(ia-s)t}dt\\
&= \left[\frac{e^{(ia-s)t}}{ia-s}\right]_0^\infty = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2}\\
&= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2},
\end{aligned}$$

and since $\mathcal{L}$ is linear Eulers formulas (6.1.1) implies that

$$\begin{aligned}
\mathcal{L}\{\cos at\}(s) &= \frac{s}{s^2+a^2},\\
\mathcal{L}\{\sin at\}(s) &= \frac{a^2}{s^2+a^2}.
\end{aligned}$$

♢

Example 6.5. Solve the initial value problem

$$\begin{cases} y'' + y &= 1, \\ y(0) = y'(0) &= 0. \end{cases}$$

Solution: Let $\mathcal{L}\{y(t)\} = Y(s)$. Then $\mathcal{L}\left\{y''(t)\right\} = s^2 Y(s)$, and if we (Laplace-) transform the equation above we get

$$\mathcal{L}\left\{y'' + y\right\} = s^2 Y(s) + Y(s) = \mathcal{L}\{1\} = \frac{1}{s},$$

i.e.

$$Y(s) = \frac{\frac{1}{s}}{1+s^2} = \frac{1}{s(1+s^2)} = \frac{1}{s} - \frac{s}{1+s^2}.$$

If we apply the inverse Laplace transform we get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{s}{1+s^2}\right\}(t) \\ &= 1 - \cos t. \end{aligned}$$

♢

Example 6.6. (The heat conduction equation)
Consider the boundary value problem

$$\begin{cases} u_t - k u_{xx} = 0, \quad t > 0, \quad x > 0, \\ u(x,0) = 0, \quad x > 0, \\ u(0,t) = 1, \quad t > 0, \end{cases}$$

where $u(x,t)$ is a bounded function ($u(x,t)$ gives the heat in the point x at the time t). Now transform the entire equation in the time variable, and let $U(x,s)$ denote the Laplace transform of $u(x,t)$. The equation $u_t - k u_{xx} = 0$ can now be written as

$$sU(x,s) - kU_{xx}(x,s) = 0,$$

and if we solve this ordinary differential equation (in the x-variable), we get

$$U(x,s) = Ae^{-\sqrt{s/k}x} + Be^{\sqrt{s/k}x},$$

where A and B are functions of s. Since we assumed u to be bounded (in both variables) the term containing $e^{\sqrt{s/k}x}$ must vanish, i.e. $B = 0$, and we get

$$U(x,s) = A(s)e^{-\sqrt{s/k}x}.$$

The boundary condition implies $U(0,s) = \mathcal{L}\{u(0,t)\}(s) = \mathcal{L}\{1\}(s) = \frac{1}{s}$, but we see that $U(0,s) = A(s)$ so $A(s) = \frac{1}{s}$ and

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s/k}x}.$$

To find u we must now apply the inverse transform on U. For this purpose it is convenient to use a table, and using Appendix 1, p. 90 we see that

$$\boxed{u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right),}$$

where erfc is the complementary error function (erf),

$$\begin{aligned} \operatorname{erfc}(t) &= 1 - \operatorname{erf}(t), \\ \operatorname{erfc}(t) &= \frac{2}{\sqrt{\pi}} \int_0^t e^{-z^2} dz. \end{aligned}$$

6.3. The Fourier Transform

The counterpart of Fourier series for functions $f(t)$ defined on $\mathbb{R}$ is the Fourier transform, $\mathcal{F}\{f\}$, which we define as

$$\mathcal{F} : f(t) \mapsto \hat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,$$

for functions $f(t)$ such that the integral converges. We also have an inverse transform

$$\mathcal{F}^{-1} : \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

REMARK 11. We can still interpret the formula as if we reconstruct the signal $f(t)$ as a sum of waves (basis functions) $e^{i\omega t}$, with amplitudes $\hat{f}(\omega)$.

REMARK 12. In applications it is customary to find the inverse transform using appropriate tables (see Appendix 2, p. 91).

In the same manner as for the Laplace transform we can derive a number of useful general properties for the Fourier transform.

- **Linearity**
$$\mathcal{F}\{af(t) + bg(t)\} = a\mathcal{F}\{f(t)\} + b\mathcal{F}\{g(t)\}.$$
- **Differentiation**
$$\begin{aligned} \mathcal{F}\{f'(t)\} &= i\omega \mathcal{F}\{f(t)\}, \\ \mathcal{F}\{f''(t)\} &= (i\omega)^2 \mathcal{F}\{f(t)\}, \\ &\vdots \\ \mathcal{F}\left\{f^{(n)}(t)\right\} &= (i\omega)^n \mathcal{F}\{f(t)\}. \end{aligned}$$
- **Convolution**
$$\mathcal{F}\{f \star g\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\},$$
where the convolution over $\mathbb{R}$ is defined by
$$f \star g = \int_{-\infty}^{\infty} f(t-u) g(u) du.$$
- **Frequency modulation**
$$\mathcal{F}\left\{e^{iat} f(t)\right\}(\omega) = \mathcal{F}\{f(t)\}(\omega - a) = \hat{f}(\omega - a).$$

- **Time delay**

$$\mathcal{F}\{f(t-a)\} = e^{-i\omega a}\hat{f}(\omega).$$

Example 6.7. Let $f(t) = \theta(t)e^{-t}$, ($\theta(t)$ is defined as on p. 46) then

$$\begin{aligned}\mathcal{F}\left\{\theta(t)e^{-t}\right\}(\omega) &= \int_{-\infty}^{\infty} \theta(t)e^{-t}e^{-i\omega t}dt \\ &= \int_{0}^{\infty} e^{-(1+i\omega)t}dt \\ &= \left[\frac{-e^{-(1+i\omega)t}}{1+i\omega}\right]_0^{\infty} = \frac{1}{1+i\omega}.\end{aligned}$$

I.e. $\hat{f}(\omega) = \dfrac{1}{1+i\omega}$.

◇

Example 6.8. (Heat conduction equation with an initial temperature distribution)

Assume that we have an infinitely long rod with temperature distribution in the point x at the time t given by $u(x,t)$, $x \in \mathbb{R}, t \geq 0$. Assume also that at the initial time $t = 0$ the temperature is distributed according to the function $f(x)$, i.e. $u(x,0) = f(x)$. To determine u we must solve the following initial value problem.

$$\begin{cases} u_t' - ku_{xx}'' = 0, & -\infty < x < \infty, t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases} \tag{6.3.1}$$

Solution: By using the Fourier transform in the same way as the Laplace transform in Example 6.6 we get (after some calculations) that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(z)e^{-(x-z)^2/4kt}dz.$$

◇

REMARK 13. The function $G(y,t) = \dfrac{1}{\sqrt{4\pi kt}}e^{-(x-z)^2/4kt}$ is the so called Green's function or the *unit impulse solution* to the following problem:

$$\begin{cases} G_t' - kG_{xx}'' &= 0, \\ G(x,0) &= \delta_y(x). \end{cases}$$

Here $\delta_y(x)$ is the Dirac delta function (Paul Dirac), which is usually characterized by the property that

$$\int_{-\infty}^{\infty} g(x)\delta_y(x)dx = g(y),$$

or alternatively formulated

$$g \star \delta_y(u) = g(u-y).$$

Green's method: The solution to 6.3.1 is given by

$$u = f \star G.$$

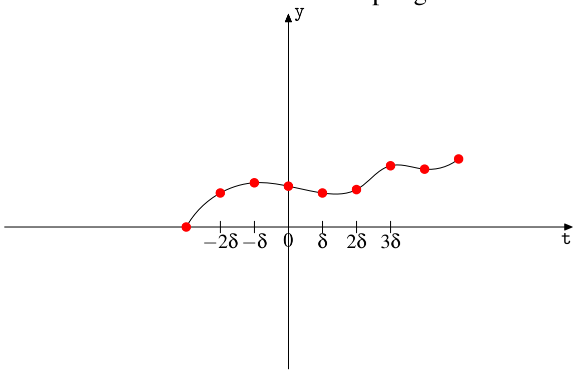
Observe that $\delta_y(x)$ is not a function strictly speaking, but a *distribution*. If $y = 0$ we simply write $\delta_0(x) = \delta(x)$. I connection with applications $\delta_y(x)$ is usually called a *unit impulse* (in the point $x = y$). When considering a physical system, the occurrence of $\delta(t)$ should be viewed as that the system is subjected to a short (momentary) force. (For example if you hit a pendulum with a hammer at the time 0 the system will be described by an equation of the type $m\ddot{y} + a\dot{y} + by = c\delta(t)$.)

Sampling

Sampling here means that we reconstruct a continuous function from a set of discrete (measured/sampled) function values.

$$\mathcal{S} : f(t) \to \{f(n\delta)\}, \delta \text{ is the length of the sampling interval.}$$

FIGURE 6.3.1. Sampling



DEFINITION 6.1. A function $f(t)$ is said to be *band limited* if the Fourier transform of f, $\mathcal{F}(f)$ only contains frequencies in a bounded interval, i.e. if $\hat{f}(\omega) = 0$ for $|\omega| \geq c$ for some constant c. (The counterpart for periodic functions is of course that the Fourier series transform is a finite sum.)

THEOREM. *The sampling theorem*

A continuous band limited signal $f(t)$ can be uniquely reconstructed from its values in a finite number of uniformly distributed points (sampling points) if the distance between two neighboring points is at most $\frac{\pi}{c}$. In this case we have:

$$\mathcal{S}^{-1} : f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{c}\right) \frac{\sin(ct - k\pi)}{ct - k\pi}.$$

(Here the sampling is performed over the points $x_k = \frac{k\pi}{c}$.)

REMARK 14. In connection with the sampling theorem we should also mention two other discrete Fourier transforms:

- The Discrete Fourier Transform (DFT).
- The Fast Fourier Transform (FFT).

These transforms are very useful in many practical applications, but we do not have the time to go into more details concerning these in this short introduction (in short one can say that practically the entire information society of today relies on the FFT). Some references:

- Mathematics of the DFT. A good and extensive online-book on DFT and applications, http://ccrma-www.stanford.edu/~jos/r320/.

- Fourier Transforms, DFTs, and FFTs. Another extensive text on mainly DFT and FFT with examples and applications,
  http://www.me.psu.edu/me82/Learning/FFT/FFT.html.

6.4. The Z-transform

Consider discrete signals, $\{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, \ldots\}$, or $\{x_n\}_{n=-\infty}^{\infty} = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$. The notation

$$\left\{1, 2, \overset{\downarrow}{5}, 6, -1, \ldots\right\}$$

implies that $x_0 = 5$. The Z-transform of the sequence $\{x_n\}$ is defined by

$$\begin{aligned} \mathcal{Z}: \quad \{x_n\} \quad &\to X(z) = \sum_{n=0}^{\infty} x_n z^{-n}, \\ \mathcal{Z}^{-1}: \quad \mathcal{Z}^{-1}[X] \quad &= \{x_n\}_{n=0}^{\infty}. \end{aligned}$$

The Z-transform can be considered as a discrete version of the Laplace transform and therefore it is not surprising that similar general properties hold. For example we have:

- **Linearity**
$$\mathcal{Z}[a\{x_n\} + b\{y_n\}] = a\mathcal{Z}[\{x_n\}] + b\mathcal{Z}[\{y_n\}].$$
- **Damping**
$$\mathcal{Z}[\{a^n x_n\}] = X\left(\frac{z}{a}\right).$$
- **Convolution**
$$\mathcal{Z}[\{x_n\} \star \{y_n\}] = \mathcal{Z}[\{x_n\}] \cdot b\mathcal{Z}[\{y_n\}],$$
  where (the discrete) convolution of two sequences is defined by
$$\{x_n\} \star \{y_n\} = \{z_n\}, \text{ with } z_n = \sum_{k=1}^{n} x_{n-k} y_k, \; n = 0, 1, 2, \ldots.$$
- **Differentiation**
$$X'(z) = \mathcal{Z}[\{0, 0, -x_1, -2x_2, -3x_3, \ldots\}].$$
- **Forward shift**
$$\begin{aligned} \mathcal{Z}[\{0, x_0, x_1, x_2, x_3, \ldots\}] &= z^{-1} X(z), \\ \mathcal{Z}[\{0, 0, x_0, x_1, x_2, x_3, \ldots\}] &= z^{-2} X(z), \text{ etc.} \end{aligned}$$
- **Backward shift**
$$\begin{aligned} \mathcal{Z}\left[\{x_0, \overset{\downarrow}{x_1}, x_2, x_3, \ldots\}\right] &= zX(z) - x_0 z, \\ \mathcal{Z}\left[\{x_0, x_1, \overset{\downarrow}{x_2}, x_3, \ldots\}\right] &= z^2 X(z) - x_0 z^2 - x_1 z, \text{ etc.} \end{aligned}$$

When comparing with the formulas for the Laplace transform we se that the forward shift corresponds to time delay and backward shift corresponds to differentiation in the continuous case. Since the shift operations might feel a little different as compared to their continuous counterparts we prove the second last equality:

$$\begin{aligned} \mathcal{Z}\left[\{x_0, \overset{\downarrow}{x_1}, x_2, x_3, \ldots\}\right] &= x_1 + x_2 z^{-1} + x_3 z^{-2} + \cdots \\ &= x_0 z + x_1 + x_2 z^{-1} + x_3 z^{-2} + \cdots - x_0 z \\ &= zX(z) - x_0 z. \end{aligned}$$

Example 6.9. (Some examples on the Z-transform)

a) **Unit step sequence**. Let $\{\sigma_n\} = \{0, 0, \overset{\downarrow}{1}, 1, 1, \ldots\}$, then

$$\mathcal{Z}[\{\sigma_n\}] = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1},\ |z| > 1.$$

b) **Unit impulse sequence**. Let $\{\delta_n\} = \{\ldots, 0, 0, \overset{\downarrow}{1}, 0, 0, \ldots\}$, then $\{\delta_{n-2}\} = \{\ldots, 0, 0, \overset{\downarrow}{0}, 0, 1, 0, 0 \ldots\}$, and we get

$$\begin{aligned} \mathcal{Z}[\{\delta_n\}] &= 1, \\ \mathcal{Z}[\{\delta_{n-2}\}] &= \frac{1}{z^2}, \text{ etc.} \end{aligned}$$

c) **Unit ramp sequence**. Let $\{r_n\} = \{\ldots, 0, 0, \overset{\downarrow}{0}, 1, 2, \ldots\}$. Then

$$\mathcal{Z}[\{r_n\}] = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots,$$

$$\begin{aligned} \frac{1}{1-z} &= 1 + z + z^2 + z^3 + \cdots,\ |z| < 1, \\ \text{and} & \quad \text{(differentiate both sides)} \\ \frac{1}{(1-z)^2} &= 1 + 2z + 3z^2 + \cdots, \end{aligned}$$

which gives

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots,$$

and if we set $\frac{1}{z}$ instead of z here we see that

$$\mathcal{Z}[\{r_n\}] = \frac{z}{(z-1)^2},\ |z| > 1.$$

REMARK 15. The Z-transform is very useful for solving difference equations and for treating discrete linear systems.

REMARK 16. The discrete Fourier transform (DFT) that was mentioned earlier is a special case of the Z-transform with $z = e^{-2k\pi/N}$.

REMARK 17. More examples of useful transform pairs and general properties can be found in Appendix 3.

6.5. Wavelet transforms

The idea of wavelets is relatively new, but it has already shown itself to be much more effective than many other transforms, e.g. for applications in

- Signal processing, and
- Image processing.

In these cases the story begins with what is now called the *mother wavelet,* ψ. Typically the function ψ has the following properties:

* $\displaystyle\int_{-\infty}^{\infty} \psi(t)dt = 0,$

** ψ is well localized in *both* time and frequency, and in addition satisfies some further (technical) conditions.

It can then be shown that the following system

$$\left\{\psi_{j,k}(t)\right\}_{j,k=-\infty}^{\infty},$$

where

$$\psi_{j,k}(t) = 2^{j/k}\psi\left(2^j t - k\right)$$

are *translations*, *dilatations* and *normalization of* the original mother wavelet, is a (complete) orthogonal basis. A signal $f(t)$ can be reconstructed by using the usual (generalized) Fourier idea:

$$\mathcal{W}^{-1} : f(t) = \sum_{j,k=-\infty}^{\infty} \left\langle f, \psi_{j,k}\right\rangle \psi_{j,k}(t).$$

and we also have

$$\mathcal{W} : f(t) \to \left\{\left\langle f, \psi_{j,k}\right\rangle\right\}_{j,k=-\infty}^{\infty},$$

where the "Fourier coefficients" are given by the scalar products $\left\langle f, \psi_{j,k}\right\rangle = \displaystyle\int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)dt.$

REMARK 18. A problem with the Fourier series transform is that a signal $f(t)$ which is well localized in time results in an outgoing signal $\hat{f}(\omega)$ which is dispersed in the frequency range (e.g. the Fourier series for the delta function $\delta(t)$ contains all frequencies) and vice versa. The advantage with the wavelet transform is that you can "compromise" and obtain localization in both time and frequency simultaneously (at least in certain cases).

REMARK 19. In Appendix 4 we have included a motivation and illustration which makes it easier to understand the terminology and formulas above. The motivation is obtained by a natural approximation procedure, with the classical Haar wavelet as mother wavelet.

REMARK 20. The transform $\mathcal{W}$ above corresponds to the Fourier series transform, but there also exists a similar integral transform corresponding to the Fourier transform.

REMARK 21. The wavelet transforms are not so useful if you have to do all calculations by hand, but nowadays there are easily available computer programs which makes them very powerful for certain applications. The following web adresses provide information about a few such programs:

- `http://www.wavelet.org` (Wavelet Digest+search engine+links+...)
- `http://www.finah.com/` (Many practical applications)
- `http://www.tyche.math.univie.ac.at/Gabor/index.html` (Gabor analysis)
- `http://www.sm.luth.se/~grip/` (Licentiate and PhD thesis of Niklas Grip)

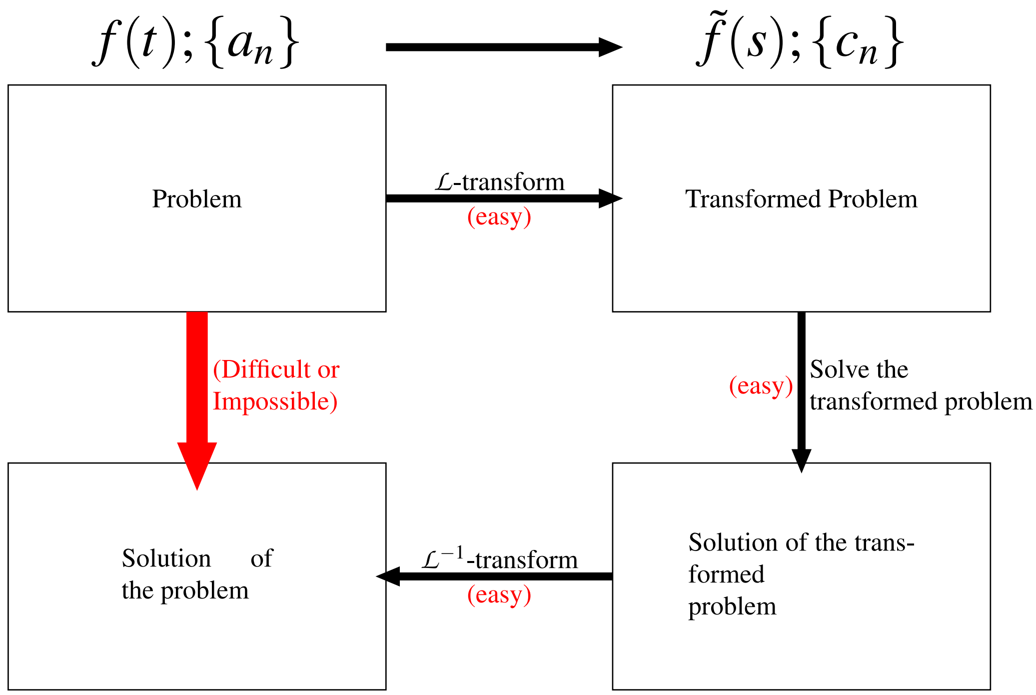
Some research groups in Sweden which are working with wavelets and applications (also industrially):

- KTH: Jan-Olov Strömberg (janolov@math.kth.se)
- Chalmers: Jöran Bergh (math.chalmers.se)
- LTU (and Uppsala): Lars-Erik Persson (larserik@sm.luth.se)

Some books on wavelets

- *Wavelets*, J. Bergh and F. Ekstedt and M. Lindberg ((1))
- *A Wavelet Tour of Signal Processing*, S.G. Mallat ((6))
- *Introduction to Wavelets and Wavelet Transforms*, A Primer, C.S. Burrus and R.A. Gopinath and H. Guo ((2))
- *Foundations of Time-Frequency Analysis*, K. Gröchenig ((4))

6.6. The General Transform Idea



When we want to solve a given problem the key to success is to chose a suitable transform for the problem in question. In this chapter we have presented some useful transforms but there are other examples in the literature. In Appendix 5 we present some further transforms (mainly taken from (3)). In most cases we have also included a formula for the inverse transform and the corresponding useful tables are also included.

6.7. Continuous Linear Systems

FIGURE 6.7.1. A schematic picture of a continuous linear system

Many linear system, e.g. in technical applications, can be described by a linear differential equation:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) = b_k x^{(k)}(t) + b_{k-1} x^{(k-1)}(t) + \cdots + b_0 x(t), \tag{6.7.1}$$

together with initial values

$$y(0) = y'(0) = \cdots = y^{(n)}(0) = 0.$$

Set $Y(s) = \mathcal{L}\{y(t)\}(s)$ and $X(s) = \mathcal{L}\{x(t)\}(s)$ and transform (6.7.1). Using the initial values we get:

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots a_0) Y(s) = (b_k s^k + b_{k-1} s^{k-1} + \cdots b_0) X(s),$$

which gives

$$\frac{Y(s)}{X(s)} = \frac{b_k s^k + b_{k-1} s^{k-1} + \cdots b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0}.$$

We define the *Transfer function*, $H(s)$, by $Y(s) = H(s)X(s)$, i.e.

$$\boxed{H(s) = \frac{b_k s^k + b_{k-1} s^{k-1} + \cdots b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0}.}$$

For every incoming signal (with transform $X(s)$) we get the corresponding solution (outgoing signal) $Y(s) = H(s)X(s)$, and if we invert the transform we see that

$$y(t) = h(t) \star x(t).$$

How do we find $H(s)$?

For a unit impulse $\delta(t)$ we have

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st} dt = e^0 = 1.$$

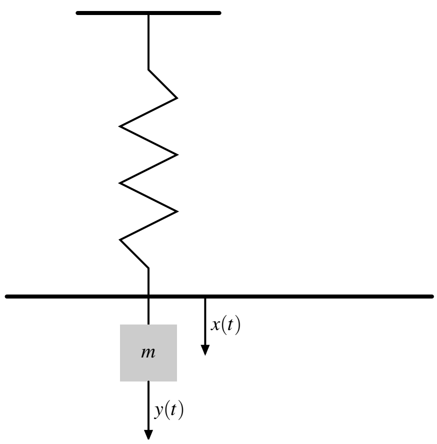
This implies that if we send in a unit impulse the system will respond in the following way:

$$\begin{cases} y(t) & = h(t) \star \delta(t) = h(t), \\ Y(s) & = H(s). \end{cases}$$

In technical applications $h(t)$ is usually called the *unit impulse solution*.

Example 6.10. (Driven harmonic oscillator)

FIGURE 6.7.2. Hanging spring



We consider the system illustrated in Figure 6.7.2, i.e. a weight m which is attached to the end of vertically suspended spring. The weight has an equilibrium point relative to a moving reference system (e.g. the point of attachment for the spring), and the distance from this equilibrium point is denoted by $y(t)$. The movement of the reference system (relative to some absolute reference system) is denoted by $x(t)$.

(A concrete example of such a system with a moving reference system is obtained attaching the spring to a wooden board and then move that board up and down.)

It can be shown that the system can be described by the following linear differential equation:

$$m\ddot{y}(t)+c\dot{y}(t)+ky(t)=c\dot{x}(t)+ax(t).$$

If we apply the Laplace transform to both sides of this equation we get

$$\left(ms^2+cs+k\right)Y(s)=(cs+a)X(s),$$

and the transfer function is

$$H(s)=\frac{cs+a}{ms^2+cs+k}.$$

Suppose, for example, that we have the incoming signal $x(t)=\sin\omega t$ and that $m=1.00$kg, $c=0$, $k=a=1000$N/m and $\omega=2\pi$. Then $X(s)=\dfrac{\omega}{s^2+\omega^2}$,

$$Y(s)=H(s)X(s)=\frac{cs+a}{ms^2+cs+k}\cdot\frac{\omega}{s^2+\omega^2},$$

and if we insert the values we get

$$Y(s)=\frac{1000}{s^2+1000}\cdot\frac{2\pi}{s^2+4\pi^2}=\frac{D}{s^2+4\pi^2}-\frac{D}{s^2+1000},$$

where $D=\dfrac{2000\pi}{1000-4\pi^2}$. Thus

$$y(t)=\frac{D}{2\pi}\sin 2\pi t-\frac{D}{\sqrt{1000}}\sin\sqrt{1000}t\approx 1.04\sin 6.28t-0.207\sin 31.6t.$$

◊

It is sometimes also useful to compute the unit step solution, i.e. the reaction of the system on the incoming signal

$$\theta(t)=\begin{cases}1, & t>0,\\ 0, & t\le 0.\end{cases}$$

We know that $\mathcal{L}\{\theta(t)\}=\dfrac{1}{s}$ hence $Y(s)=\dfrac{1}{s}\cdot H(s)$.

Example 6.11. A system has the transfer function

$$H(s)=\frac{3}{(s+1)(s+3)}.$$

Compute the unit step solution!

Solution: We know that

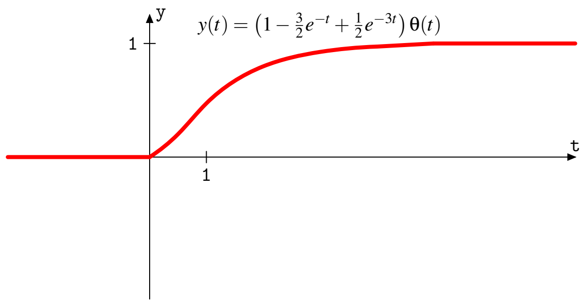
$$Y(s)=\frac{1}{s}H(s)=\frac{3}{s(s+1)(s+3)}=\frac{1}{s}-\frac{3}{2(s+1)}+\frac{1}{2(s+3)},$$

and hence $y(t)=1-\dfrac{3}{2}e^{-t}+\dfrac{1}{2}e^{-3t}$ for $y\ge 0$ (and $y(t)=0$ for $y<0$), i.e.

$$y(t)=\left(1-\frac{3}{2}e^{-t}+\frac{1}{2}e^{-3t}\right)\theta(t),$$

see Fig. 6.7.3.

FIGURE 6.7.3.



◇

6.8. Discrete Linear Systems

FIGURE 6.8.1. A schematic image of a discrete linear system

A discrete linear system can be described by a linear difference equation:

(6.8.1) $$a_0 y_n + a_1 y_{n-1} + \cdots + a_m y_{n-m} = b_0 x_n + b_1 x_{n-1} + \cdots + b_k x_{n-k},$$

alternatively this equation can be formulated as

$$\{a_k\} \star \{y_k\} = \{b_k\} \star \{x_k\}.$$

Let $Y(s) = \mathcal{Z}[\{y_n\}](z)$, and $X(s) = \mathcal{Z}[\{x_n\}](z)$. A Z-transform of (6.8.1) gives the equation

$$\left(a_0 + a_1 \frac{1}{z} + \cdots + a_m \frac{1}{z^m}\right) Y(z) = \left(b_0 + b_1 \frac{1}{z} + \cdots + b_k \frac{1}{z^k}\right) X(z),$$

i.e.

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 \frac{1}{z} + \cdots + b_k \frac{1}{z^k}}{a_0 + a_1 \frac{1}{z} + \cdots + a_m \frac{1}{z^m}},$$

and in the same way as before we can define a transfer function, $H(z)$, by

$$\boxed{H(z) = \frac{b_0 + b_1 \frac{1}{z} + \cdots + b_k \frac{1}{z^k}}{a_0 + a_1 \frac{1}{z} + \cdots + a_m \frac{1}{z^m}}.}$$

For every incoming signal (with Z-transform $X(z)$) we get the solution (outgoing signal)

$$Y(z) = H(z)X(z),$$

which gives

$$\{y_n\} = \{h_n\} \star \{x_n\}.$$

How do we find $H(z)$?

For the unit impulse sequence, $\{\delta_n\}$, we have $\mathcal{Z}[\{y_n\}] = 1 + 0 \cdot \frac{1}{z} + \cdots = 1$, which implies that the system will respond in the following way:

$$\{y_n\} = \{h_n\} \star \{\delta_n\} = \{h_n\},$$

i.e.

$$Y(z) = H(z).$$

In technical applications $\{h_n\}$ is called the unit impulse response.

Example 6.12. A linear discrete system has the transfer function $H(z) = \frac{1}{z+0.8}$. Compute the unit step response!

Solution: The unit step sequence is $\{\sigma_n\} = \{1, 1, 1, \ldots\}$, and we have

$$X(z) = \mathcal{Z}[\{\sigma_n\}] = \frac{z}{z-1},$$

and thus we get

$$Y(z) = H(z)X(z) = \frac{z}{(z-1)(z+0.8)} = \frac{5z}{9}\left[\frac{1}{(z-1)} - \frac{1}{(z+0.8)}\right].$$

The inverse transform gives the answer, $\mathcal{Z}^{-1}[Y(z)] = \{y_n\}$, where

$$y_n = \frac{5}{9}\left(1 - (-0.8)^n\right).$$

6.9. Further Examples

Example 6.13. Compute the integral $\int_0^\infty \frac{\sin ax}{x(1+x^2)} dx$ for $a > 0$.

Solution: Consider

$$f(t) = \int_0^\infty \frac{\sin tx}{x(1+x^2)} dx, \, t > 0$$

and its Laplace transform

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \tilde{f}(s) &= \int_0^\infty \left(\int_0^\infty \frac{\sin tx}{x(1+x^2)} dx\right) e^{-st} dt \\
&= \int_0^\infty \left(\int_0^\infty \sin(tx) e^{-st} dt\right) \frac{1}{x(1+x^2)} dx \\
&= \int_0^\infty \mathcal{L}(\sin tx)(s) \frac{1}{x(1+x^2)} dx \\
&= \int_0^\infty \frac{x}{(x^2+s^2)} \frac{1}{x(1+x^2)} dx \\
&= \frac{1}{s^2-1} \int_0^\infty \frac{1}{1+x^2} - \frac{1}{x^2+s^2} dx \\
&= \frac{1}{s^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2}\frac{1}{s}\right) = \frac{\pi}{2}\left(\frac{1}{s} - \frac{1}{s+1}\right).
\end{aligned}$$

By applying the inverse transform we see that

$$f(t) = \frac{\pi}{2}\left(1 - e^{-t}\right),$$

i.e.

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx = \frac{\pi}{2}\left(1-e^{-a}\right), a>0.$$

♢

Example 6.14. (The Dirichlet problem (Lejeune Dirichlet) for a half-plane)
Solve

$$\begin{cases} u''_{xx}+u''_{yy} & = 0,\ -\infty < x < \infty,\ y \geq 0, \\ u(x,0) & = f(x), \\ u(x,y) \to 0, & \text{when } |x| \to \infty,\ y \to \infty. \end{cases} \tag{6.9.1}$$

Solution: We start by applying the Fourier transform (with respect to x) to u. We denote this operation with $\mathcal{F}_x\{u\} = \mathcal{F}\{x \mapsto u(x,y)\}$ and we get

$$U = U(\omega,y) = \mathcal{F}_x\{u\}(\omega) = \int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}dx,$$

and (6.9.1) is then transformed into

$$\begin{cases} \dfrac{d^2U}{dy^2} - \omega^2 U & = 0, \\ U(\omega,0) & = \hat{f}(\omega), \\ U(\omega,y) \to 0, & \text{when } y \to \infty. \end{cases}$$

The solution to this transformed problem is given by

$$U(\omega,y) = \hat{f}(\omega)e^{-|\omega|y}.$$

If we use the convolution property ($\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$) we see that

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}(U) = \mathcal{F}^{-1}\left\{\hat{f}(\omega)e^{-|\omega|y}\right\} = \mathcal{F}^{-1}\{\mathcal{F}(f)\mathcal{F}(g_y)\} \\ & \int_{-\infty}^{\infty} f(z)g_y(x-z)dz, \end{aligned}$$

where $g_y(x)$ is the inverse Fourier transform of $e^{-|\omega|y}$, i.e.

$$g_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Hence the wanted solution is

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(z)}{(x-z)^2+y^2}dz,\ y>0.$$

(This is the famous Poisson integral formula).

♢

6.10. Exercises

6.1. [A] a) Compute the inverse Laplace transform of

$$F(s) = e^{-2s}\frac{1}{s^2+8s+15}.$$

b) Find the unit step response to a system with the transfer function

$$H(s) = \frac{3}{(s+1)(s+3)}.$$

6.2.* Use the Laplace transform to solve:

$$\begin{cases} u_t'(x,t) & = u_{xx}''(x,t),\, 0 \le x < 1,\, t > 0, \\ u(0,t) & = u(1,t) = 1,\, t > 0, \\ u(x,0) & = 1 + \sin \pi x,\, 0 < x < 1. \end{cases}$$

6.3. [A] Use the Laplace transform to solve:

$$y'' + 2y' + 2y = u(t),\, y(0) = y'(0) = 0.$$

a) When $u(t) = \theta(t)$,

b) when $u(t) = \rho(t) = \begin{cases} 0, & t \le 0, \\ 1 & t > 0. \end{cases}$

6.4. Use Fourier series to solve:

$$\begin{cases} u_{tt}'(x,t) & = u_{xx}''(x,t),\, 0 < x < 1,\, t > 0, \\ u(0,t) & = u(1,t) = 0,\, t > 0, \\ u(x,0) & = \sin \pi x,\, u_t'(x,0) = \sin 3\pi x,\, 0 < x < 1. \end{cases}$$

6.5. [A] Compute the Fourier transform of the signal

$$f(t) = \theta(t-3)e^{-(t-3)}.$$

6.6.*

a) Prove the convolution formula, $\mathcal{F}\{f \star g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$, for the Fourier transform.

b) Define $f(t) = \theta(t)e^{-t}$, let $f_1(t) = f(t)$ and for $n \ge 1$ let $f_n(t) = (f_{n-1} \star f)(t)$. Compute $f_n(t)$.

6.7. [A] Compute the Fourier transform, $\mathcal{F}(\omega)$, of

$$f(t) = \begin{cases} \sin \omega_0 t, & |t| \le a, \\ 0 & \text{else.} \end{cases}$$

6.8. Compute the Fourier transform, $\mathcal{F}(\omega)$, of

$$f(t)=\begin{cases}\cos\omega_0 t, & |t|\le a,\\ 0 & \text{else.}\end{cases}$$

6.9. [A] Solve the following difference equation

$$y(n+2)-y(n+1)-2y(n)=0,\ y(0)=2,\ y(1)=1.$$

6.10. Determine the sequence $y(n)$, $n\ge 0$ which has the Z-transform $Y(z)=\dfrac{1}{z^2+1}$.

6.11. [A] Let $f(x)=e^{-|x|}$ and compute the convolution product $(f\star f)(x)$.

6.12. Use the function $e^{-|t|}$ to

a) Compute the Fourier transform of $f(t)=\dfrac{1}{1+t^2}$.

b) Compute the Fourier transform of $g(t)=\dfrac{\alpha}{\alpha^2+t^2}$, $\alpha>0$.

c) Compute the Fourier transform of $h(t)=\dfrac{t^2-\alpha^2}{(\alpha^2+t^2)^2}$, $\alpha\neq 0$.

6.13. [A] Use the Laplace transform to solve the following system of differential equations

$$\begin{cases}x'-2x+3y &=0\\ y'-y+2x &=0\end{cases},\qquad \begin{cases}x(0) &=8\\ y(0) &=3.\end{cases}$$

6.14.

a) Define the Haar-scaling function φ and the Haar-wavelet function ψ.

b) Illustrate $\psi(t-2),\psi(4t),\psi(4t-1),\psi(4t-3)$ and $2\psi(4t-2)$ in the ty-plane.

c) Explain how a signal $f(t)$ can be represented by a system of basis functions constructed by translating, dilating and normalizing the Haar wavelet.

6.15. [A] A continuous system has the transfer function

$$H(s)=\frac{1}{1+sT}.$$

Compute the response, $y(t)$ to the signal $x(t)=\sin\omega t$.

6.16.* A discrete linear system has the transfer function

$$H(z)=\frac{1}{2z+1}.$$

Compute the unit impulse response.

6.17. [A] A discrete linear system has the unit impulse answer $\{0.7^n\}$. Compute the system's response to the signal $\{a^n\}$, $a\neq 0.7$.

6.18. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\sum_{-\infty}^{\infty}\hat{f}(n)$ is absolutely convergent, and that there is a continuous function $g(x)=\sum_{n=-\infty}^{\infty} f(2\pi n+x)$, $x\in[-\pi,\pi]$.

a) Show that $g(x)$ has the period 2π.

b) Compute the Fourier series for $g(x)$ and use this to show the following formula (the Poisson summation formula):

$$\sum_{n=-\infty}^{\infty}\hat{f}(n)=2\pi\sum_{n=-\infty}^{\infty} f(2\pi n).$$

c) Use the Fourier series for g from b) to show that if $f(x)=0$ for $|x|\geq\pi$ then we have the following formula

$$f(x)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\sum_{m=-\infty}^{\infty}\hat{f}(m)\,e^{imx}, & |x|\leq\pi,\\ 0, & |x|\geq\pi.\end{cases}$$

6.19. Use the previous exercise to show a version of the Sampling theorem. Suppose that $f:\mathbb{R}\to\mathbb{C}$ has a Fourier transform and is band-limited, i.e. $\hat{f}(\omega)=0$ for $|\omega|\geq c$. Show that f is uniquely determined by its values at (for example) the sequence $\frac{k\pi}{c}$, $k\in\mathbb{Z}$ according to the following formula

$$f(x)=\sum_{-\infty}^{\infty} f\left(\frac{k\pi}{c}\right)\frac{1}{cx-m\pi}\sin(ct-n\pi).$$

6.20. [A] The dispersion of smoke from a smoke pipe with the height h as the wind direction and and wind speed is constant can be modeled by the following equation

$$v\frac{\partial c}{\partial x}=d\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial z^2}\right),$$

where $c(x,z)$ is the concentration of smoke at the height z counted from the base of the pipe and the distance x from the pipe in the direction of the wind. d is a diffusion coefficient and v is the wind speed (in m/s). If we also assume that the rate of change of c in the x-direction is much smaller than the rate of change in the z -direction we get the simplified equation

$$\frac{\partial c}{\partial x}=k\left(\frac{\partial^2 c}{\partial z^2}\right).$$

The rate of change in the concentration at ground level and infinitely high up can be viewed as negligible which gives us the boundary values

$$\frac{\partial c}{\partial z}(x,0)=\lim_{z\to\infty}\frac{\partial c}{\partial z}(x,z)=0.$$

The concentration of smoke can also be neglected infinitely far away in the x-direction. At the location of the pipe the concentration is 0 except for at the height h where the smoke drifts out of it with a flow rate $q\text{kgm}^{-2}\text{s}^{-1}$. Thus we also get the boundary values:

$$c(0,z)=\frac{q}{v}\delta(z-h),\ \lim_{x\to\infty}c(x,z)=0.$$

a) Rewrite the problem using dimensionless quantities.

b) Use the Laplace transform to find the concentration at ground level, $c(x,0)$. (Hint: split into two cases, $\bar{z} \gtrless 1$ and observe that the derivative of the Laplace transform of c is not continuous everywhere.)

c) At which range from the pipe is the concentration at ground level highest?

A-1. General Properties of the Laplace Transform:

TABLE 1. General Properties of the Laplace Transform

	$f(t)$	$F(s)=\mathcal{L}[f(t)](s)$
Definition	$f(t)$	$\int_0^\infty f(t)e^{-st}\,dt$
Inverse	$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds$	$F(s)$
Linearity	$af(t)+bg(t)$	$a\mathcal{L}[f(t)](s)+b\mathcal{L}[g(t)](s)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$
Sign change	$f(-t)$	$F(-s)$
Time delay	$f(t-a)\theta(t-a)$	$e^{-as}F(s)$
Ampl. modulation	$f(t)\cos\Omega t$	$\frac{1}{2}\left(F\left(s-i\Omega\right)+F\left(s+i\Omega\right)\right)$
Damping	$e^{-at}f(t)$	$F(s+a)$
Convolution	$f\star g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	$\mathcal{L}[f(t)]\mathcal{L}[g(t)]$
Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
Differentiation	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
Transform pairs		
Constant	1	$s^{-1},\ s>0$
Exponential	e^{at}	$\frac{1}{s-a},\ s>a$
Power	$t^n,\ n\in\mathbb{Z}^+$	$\frac{n!}{s^{n+1}},\ s>0$
Trig.	$\sin at$ and $\cos at$	$\frac{a}{s^2+a^2}$, and $\frac{s}{s^2+a^2},\ s>0$
Hyp. trig.	$\sinh at$ and $\cosh at$	$\frac{a}{s^2-a^2}$, and $\frac{s}{s^2-a^2},\ s>\lvert a\rvert$
Exp.×trig.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\ s>a$
Exp.×trig.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s>a$
Exp.×power	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}},\ s>a$
Heaviside's function	$\theta(t-a)$	$s^{-1}e^{-as},\ s>0$
Delta function	$\delta(t-a)$	e^{-as}
Error function	$\operatorname{erf}\sqrt{t}=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-z^2}dz$	$\frac{1}{s\sqrt{1+s}},\ s>0$
Normal dist./Gaussian.	$\frac{1}{\sqrt{t}}e^{-\frac{a^2}{4t}}$	$\sqrt{\frac{\pi}{s}}e^{-a\sqrt{s}},\ s>0$
Compl. Erf.	$\operatorname{erfc}\frac{a}{2\sqrt{t}}=1-\operatorname{erf}\frac{a}{2\sqrt{t}}$	$\frac{1}{s}e^{-a\sqrt{s}},\ s>0$
$\frac{1}{t}$×Normal dist.	$\frac{a}{2t^{3/2}}e^{-\frac{a^2}{4t}}$	$\sqrt{\pi}e^{-a\sqrt{s}},\ s>0$

A-2. General Properties of the Fourier Transform

TABLE 2. General Properties of the Fourier Transform

	$f(t)$	$\hat{f}(\omega)$
Definition	$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
Inverse	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}d\omega$	$\hat{f}(\omega)$
Linearity	$af(t)+bg(t)$	$a\hat{f}(\omega)+b\hat{g}(\omega)$
Scaling	$f(at),\, a \neq 0$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\omega}{a}\right)$
Sign change	$f(-t)$	$\hat{f}(-\omega)$
Complex conjugation	$\overline{f(t)}$	$\overline{\hat{f}(-\omega)}$
Time delay	$f(t-T)$	$e^{-i\omega T}\hat{f}(\omega)$
Freq. translations	$e^{i\Omega t}f(t)$	$\hat{f}(\omega-\Omega)$
Ampl. modulation	$f(t)\cos\Omega t$	$\frac{1}{2}\left(\hat{f}(\omega-\Omega)+\hat{f}(\omega+\Omega)\right)$
Ampl. modulation	$f(t)\sin\Omega t$	$\frac{1}{2i}\left(\hat{f}(\omega-\Omega)-\hat{f}(\omega+\Omega)\right)$
Symmetry	$\hat{f}(t)$	$2\pi f(-\omega)$
Time differentiation	$f^{(n)}(t)$	$(i\omega)^n\hat{f}(\omega)$
Freq. differentiation	$(-it)^n f(t)$	$\hat{f}^{(n)}(\omega)$
Time convolution	$f(t)\star g(t)$	$\hat{f}(\omega)\hat{g}(\omega)$
Freq. convolution	$f(t)g(t)$	$\frac{1}{2\pi}\hat{f}(\omega)\star\hat{g}(\omega)$
Transform pairs		
Delta function	$\delta(t)$	1
Derivative av Delta fn..	$\delta^{(n)}(t)$	$(i\omega)^n$
Exponential	$\theta(t)e^{-at}$	$\frac{1}{a+i\omega},\, a>0$
Exponential	$(1-\theta(t))\,e^{-at}$	$\frac{1}{a-i\omega},\, a>0$
Exponential	$e^{-a\lvert t\rvert}, a>0$	$\frac{2a}{a^2+\omega^2}$
Heaviside's function	$\theta(t)$	$\pi\delta(\omega)+\frac{1}{i\omega}$
Constant	1	$2\pi\delta(\omega)$
Filtering (sinc)	$\frac{\sin\Omega t}{\pi t}$	$\theta(\omega+\Omega)-\theta(\omega-\Omega)$
Normal dist./Gaussian.	$\frac{1}{\sqrt{4\pi A}}e^{-t^2/(4A)}$	$e^{-A\omega^2},\, A>0$

A-3. General Properties of the Z-transform

TABLE 3. General Properties of the Z-transform

	$\{x_n\}_{n=0}^{\infty}$	$X(z) = \mathcal{Z}[\{x_n\}](z)$
Definition	x_n	$\sum_{n=0}^{\infty} x_n z^{-n}$
Linearity	$a\{x_n\} + b\{y_n\}$	$a\mathcal{Z}[\{x_n\}] + b\mathcal{Z}[\{y_n\}]$
Damping	$a^{-n}x_n$	$X(az)$, $a > 0$
	nx_n	$-zX'(z)$
Differentiation	$(1-n)x_{n-1}\sigma_{n-1}$	$X'(z)$
Convolution	$\{x_n\} \star \{y_n\}$	$X(z)Y(z)$
Forward translation	$x_{n-k}\sigma_{n-k}$ $(k \geq 0)$	$z^{-k}X(z)$
Backward translation	x_{n+k} $(k \geq 0)$	$z^k X(z) - \sum_{j=0}^{k-1} x_j z^{k-j}$
Transform pairs		
Unit step	σ_n	$\frac{z}{z-1}$
Unit pulse	δ_n	1
Delayed unit pulse	δ_{n-k}	z^{-k}
Exponential	a^n	$\frac{z}{z-a}$
Ramp function	$r_n = n\sigma_n$	$\frac{z}{(z-1)^2}$
Sine	$\sin n\theta$	$\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$
Damped sine	$a^n \sin n\theta$	$\frac{z\sin\theta}{z^2 - 2za\cos\theta + a^2}$
Cosine	$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$
Damped cosine	$a^n \cos n\theta$	$\frac{z(z-a\cos\theta)}{z^2 - 2za\cos\theta + a^2}$

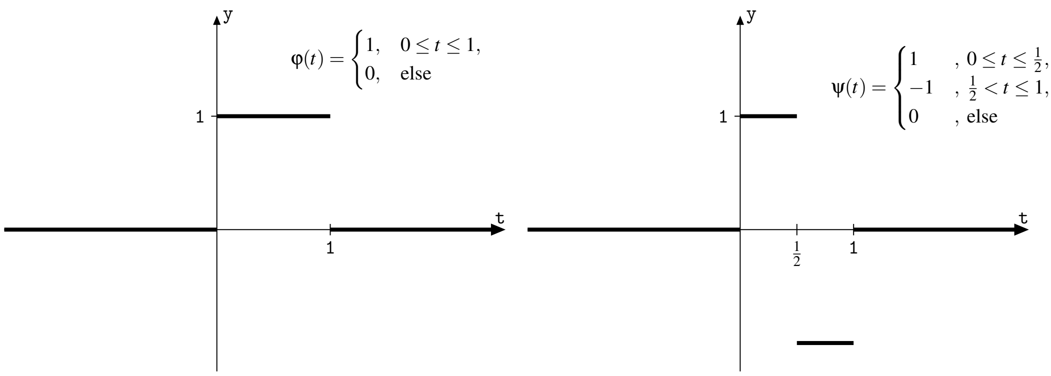
A-4. The Haar Wavelet

1.The Haar Wavelet. The mother wavelet ψ and scaling function φ are in this case very simple functions that take the values $0, 1$ and -1, and 0 and 1 (see Fig. 1.4.1):

$$\psi(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \varphi(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise}. \end{cases}$$

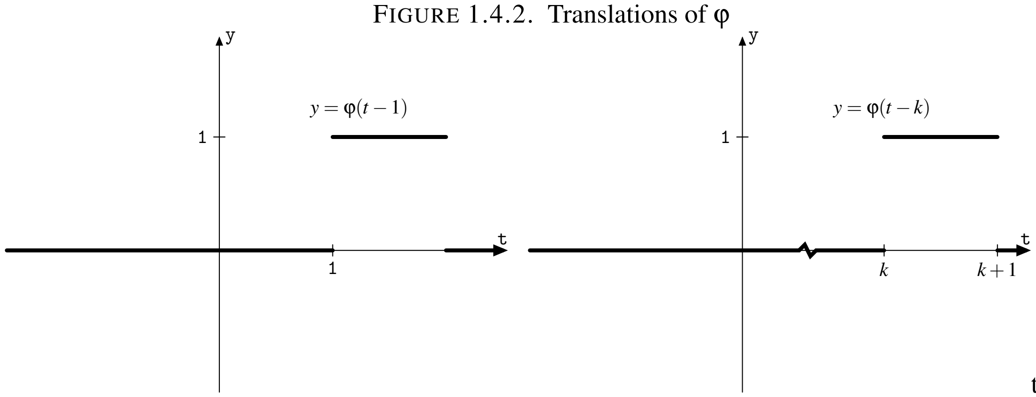
The different operations performed on the mother wavelet to construct a basis are illustrated in Figs. 1.4.2, 1.4.3, 1.4.4 and 1.4.5.

FIGURE 1.4.1. The Haar wavelet and scaling function



(a) The Haar scaling function (b) The Haar wavelet

FIGURE 1.4.2. Translations of φ



trim =

0 0 -10 0

FIGURE 1.4.3. Dilatations of φ

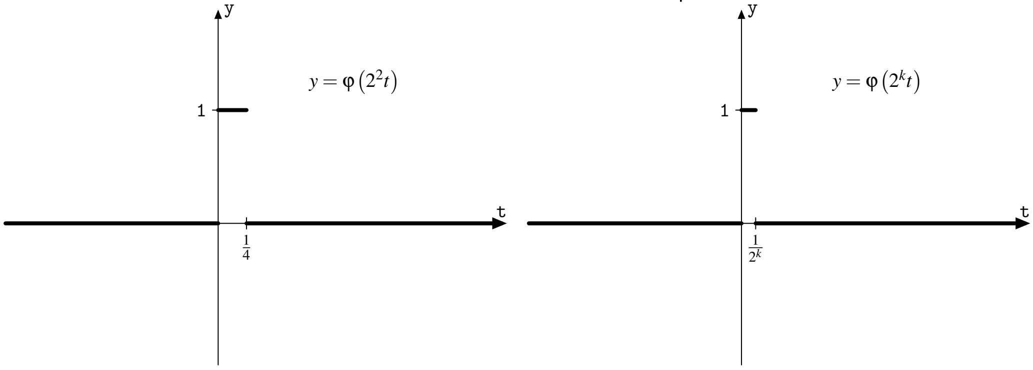


FIGURE 1.4.4. Dilatations and translations

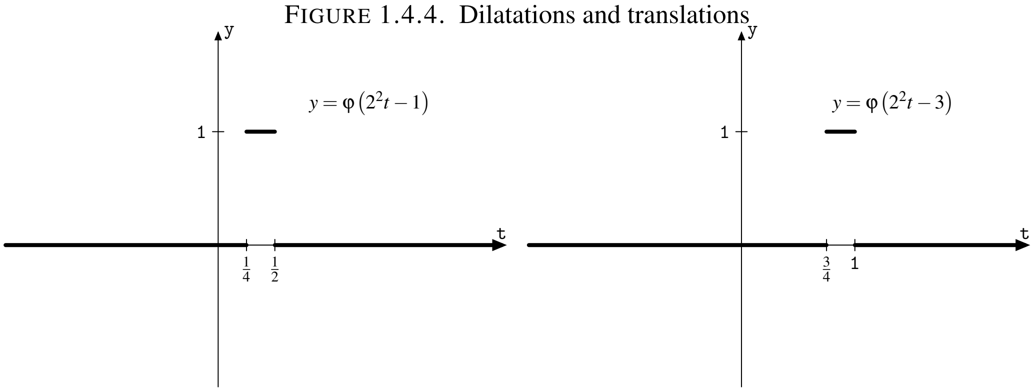
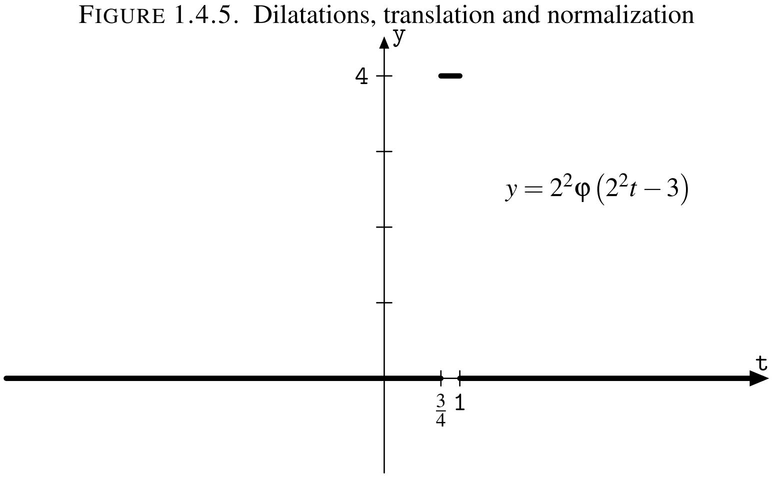


FIGURE 1.4.5. Dilatations, translation and normalization



REMARK 25. Note that we have e.g.

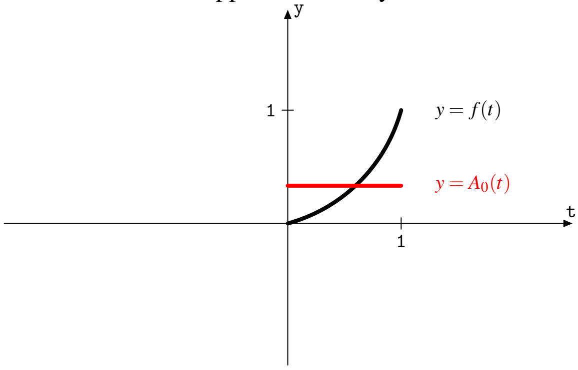
$$\begin{cases} \varphi(t) & =\varphi(2t)+\varphi(2t-1), \\ \psi(t) & =\varphi(2t)-\varphi(2t-1). \end{cases}$$

2. An Approximation Example. We will now see how to approximate a function by step functions. Observe that in the figures that illustrate the different cases we have used the function $f(t) = t^2$.

a) Approximation by the mean value (see Fig. 1.4.6):

$$f(t) \approx A_0(t) = \left(\frac{1}{1}\int_0^1 f(s)ds\right)\varphi(t).$$

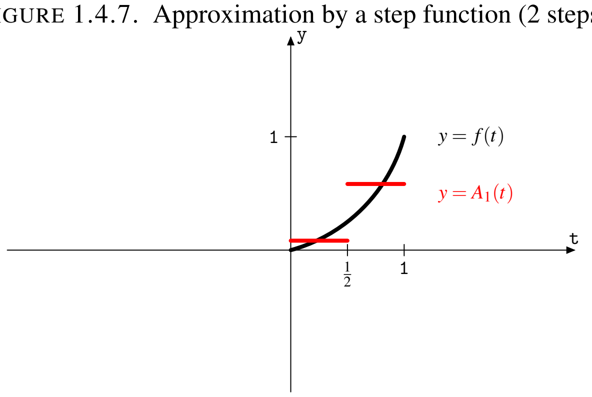
FIGURE 1.4.6. Approximation by the mean value



b) Approximation by a step function (2 steps) (see Fig. 1.4.7):

$$\begin{aligned} f(t) \approx A_1(t) &= 2\int_0^{\frac{1}{2}} f(s)ds\varphi(2t) + 2\int_{\frac{1}{2}}^1 f(s)ds\varphi(2t-1) \\ &= \int_0^1 f(s)\sqrt{2}\varphi(2s)ds\sqrt{2}\varphi(2t) + \int_0^1 f(s)\sqrt{2}\varphi(2s-1)ds\sqrt{2}\varphi(2t-1) \\ &= a_0\varphi_0(t) + a_1\varphi_1(t). \end{aligned}$$

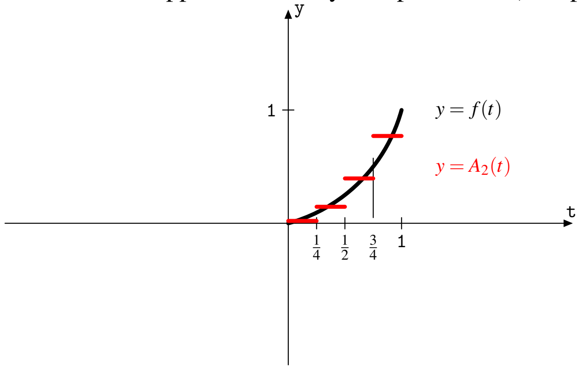
FIGURE 1.4.7. Approximation by a step function (2 steps)



c) Approximation by a step function (4 steps) (see Fig. 1.4.8):

$$
\begin{aligned}
f(t) &\approx A_2(t)\\
&= 4\int_0^{\frac{1}{4}} f(s)ds\varphi(4t) + 4\int_{\frac{1}{4}}^{\frac{1}{2}} f(s)ds\varphi(4t-1) + 4\int_{\frac{1}{2}}^{\frac{3}{4}} f(s)ds\varphi(4t-2) +\\
&\quad 4\int_{\frac{3}{4}}^{1} f(s)ds\varphi(4t-3)\\
&= \left[\int_0^1 f(s)2\varphi(4s)ds\right]2\varphi(4t) + \left[\int_0^1 f(s)2\varphi(4s-1)ds\right]2\varphi(4t-1) +\\
&\quad \left[\int_0^1 f(s)2\varphi(4s-2)ds\right]2\varphi(4t-2) + \left[\int_0^1 f(s)2\varphi(4s-3)ds\right]2\varphi(4t-3)\\
&= a_0\varphi_0(t) + a_1\varphi_1(t) + a_2\varphi_2(t) + a_3\varphi_3(t).
\end{aligned}
$$

FIGURE 1.4.8. Approximation by a step function (4 steps)



d) Approximation by a step function (2^n steps)

$$f(t) \approx \sum_{k=0}^{2^n-1} a_k\varphi_k(t),$$

where the "Fourier coefficients" are

$$a_k = \int_0^1 f(s)2^{\frac{n}{2}}\varphi(2^n s - k)\,ds,$$

and the "basis functions" are

$$\varphi_k(t) = 2^{\frac{n}{2}}\varphi(2^n t - k)\,.$$

3. Approximation by wavelets. The basic idea is that we can write $f(t)$ as:

$$
\begin{aligned}
f(t) \approx A_n(t) &= (A_n(t) - A_{n-1}(t)) + (A_{n-1}(t) - A_{n-2}(t)) + \dots\\
&\quad + (A_2(t) - A_1(t)) + (A_1(t) - A_0(t)) + A_0(t).
\end{aligned}
$$

E.g. for $n = 2$ we have

$$f(t) \approx A_2(t) = (A_2(t) - A_1(t)) + (A_1(t) - A_0(t)) + A_0(t),$$

where

$$\begin{aligned}
A_1(t)-A_0(t) &= 2\int_0^1 f(s)\varphi(2s)ds\varphi(2t)+2\int_0^1 f(s)\varphi(2s-1)ds\varphi(2t-1)- \\
&\quad \int_0^1 f(s)\varphi(s)ds\varphi(t) = [\varphi(t)=\varphi(2t)+\varphi(2t-1)] \\
&= \int_0^1 f(s)\left(\varphi(2s)-\varphi(2s-1)\right)ds\varphi(2t)-\int_0^1 f(s)\left(\varphi(2s)-\varphi(2s-1)\right)\varphi(2t-1) \\
&= \int_0^1 f(s)\psi(s)ds\psi(t),
\end{aligned}$$

where $\psi(t)$ is the Haar wavelet as defined on p. 93. Similarly one can also show that

$$A_2(t)-A_1(t)=\int_0^1 f(s)\sqrt{2}\psi(2s)ds\sqrt{2}\psi(2t)+\int_0^1 f(s)\sqrt{2}\psi(2s-1)ds\psi(2t-1).$$

By continuing in this manner we find that $f(t)$ can be approximated by $A_n(t)$, which can be expressed as

$$A_n(t)=A_0(t)+\sum_{j,k=0}^{n}\langle f,\psi_{j,k}\rangle\,\psi_{j,k}(t),$$

where

$$\psi_{j,k}(t)=2^{\frac{j}{2}}\psi\left(2^j t-k\right),$$

and

$$\langle f,\psi_{j,k}\rangle=\int_0^1 f(s)\psi_{j,k}(s)ds.$$

A-5. Additional Transforms

We present here some additional examples of transforms. For more information and applications cf.,e.g. *L. Debnath, Integral Transforms and Their Applications*, (3).

1 **The Fourier Cosine Transform**

$$\begin{aligned}
F_c : f(t) &\rightarrow \hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(\omega t)dt, \\
F_c^{-1} : &\qquad f(t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos(\omega t)d\omega.
\end{aligned}$$

2 **The Fourier Sine Transform**

$$\begin{aligned}
F_s : f(t) &\rightarrow \hat{f}_s(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(\omega t)dt, \\
F_s^{-1} : &\qquad f(t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin(\omega t)d\omega.
\end{aligned}$$

3 **The Hankel Transforms** (Defined by the Bessel functions $\mathcal{J}_n, n=0,1,\ldots$)

$$\begin{aligned}
H_n : f(r) &\rightarrow \hat{f}_n(y)=\int_0^\infty f(r)\mathcal{J}_n(yr)rdr, \\
H_n^{-1} : &\qquad f(r)=\int_0^\infty \hat{f}_n(y)\mathcal{J}_n(yr)ydy.
\end{aligned}$$

4 **The Mellin Transform**

$$M : f(x) \quad \rightarrow \quad \tilde{f}(\alpha) = \int_0^\infty x^{\alpha-1} f(x)dx,$$
$$M^{-1} : \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-\alpha}\tilde{f}(\alpha)d\alpha,$$

here α is complex and c is chosen such that the integral converges.

5 **The Hilbert Transform**

$$H : f(t) \quad \rightarrow \quad \hat{f}_H(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}dt,$$
$$H^{-1} : \qquad f(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{f}_H(x)}{x-t}dx.$$

6 **The Stieltjes Transform**

$$S : f(t) \rightarrow \tilde{f}(z) = \int_0^\infty \frac{f(z)}{t+z}dt, \ |\arg z| < \pi.$$

Remark: This operation can be inverted, but we don't get any simple integral formula as before, hence we don't write the inverse transform explicitly here.

7 **The Generalized Stieltjes Transform**

$$S_\rho : f(t) \rightarrow \tilde{f}_\rho(z) = \int_0^\infty \frac{f(t)}{(t+z)^\rho}dt, \ |\arg z| < \pi.$$

The same remark as above applies.

8 **The Legendre Transform**

$$L : f(x) \quad \rightarrow \quad \left\{\tilde{f}(n)\right\}, \tilde{f}(n) = \int_{-1}^{1} P_n(x) f(x)dx,$$
$$L^{-1} : \qquad f(x) = \sum_{n=0}^{\infty}\frac{2n+1}{2}\tilde{f}(n)P_n(x).$$

Here $P_n(x)$ is the Legendre polynomial of degree n, which we can write explicitly as

$$P_n(x) = 2^{-n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}, \ n = 0, 1, \dots,$$

and the "Fourier coefficients" are $a_n = \frac{2n+1}{2}\tilde{f}(n)$.

9 **The Jacobi Transform**

$$\mathcal{J} : \quad f(x) \rightarrow \quad \left\{f^{\alpha,\beta}(n)\right\}, f^{\alpha,\beta}(n) = \int_{-1}^{1}(1-x)^\alpha(a+x)^\beta P_n^{\alpha,\beta}(x) f(x)dx,$$
$$\mathcal{J}^{-1} : \qquad f(x) = \sum_{n=0}^{\infty}(\delta_n)^{-1} f^{\alpha,\beta}(n)P_n^{\alpha,\beta}(x).$$

Here $P_n^{\alpha,\beta}(x)$ is the Jacobi polynomial of degree n and order α, β, which can be written explicitly as

$$P_n^{\alpha,\beta}(x) = 2^{-n}\sum_{k=0}^{\infty}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k, \ n = 0, 1, \dots,$$

and the "Fourier coefficients" are $a_n = (\delta_n)^{-1} f^{\alpha,\beta}(n)$, where

$$\delta_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(n+\alpha+\beta+1)}.$$

10 **The Laguerre Transform**

$$L: \quad f(x) \to \quad \left\{\tilde{f}_\alpha(n)\right\}, \tilde{f}_\alpha(n) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) f(x) dx,$$

$$L^{-1} \qquad f(x) = \sum_{n=0}^\infty (\delta_n)^{-1} \tilde{f}_\alpha(n) L_n^\alpha(x).$$

Here $L_n^\alpha(x)$ is the Laguerre polynomial of degree$n \geq 0$ and order $\alpha > -1$, and the "Fourier coefficients" are $a_n = (\delta_n)^{-1} \tilde{f}_\alpha(n)$, where

$$\delta_n = \frac{\Gamma(n+\alpha+1)}{n!}.$$

11 **The Hermite Transform**

$$H^*: \quad f(x) \to \quad \{f_H(n)\}, f_H(n) = \int_{-\infty}^\infty e^{-x^2} H_n(x) f(x) dx,$$

$$(H^*)^{-1}: \qquad f(x) = \sum_{n=0}^\infty \delta_n^{-1} f_H(n) H_n(x).$$

Here $H_n(x)$ is the Hermite polynomial of degree n, and the "Fourier coefficients" are $a_n = \delta_n^{-1} f_H(n)$, where

$$\delta_n = n! 2^n \sqrt{\pi}.$$

REMARK 26. Observe that the transforms 8-11 are special cases of the earlier theory for generalized Fourier series (cf. Def. 6.1).

A-6. Partial Fraction Decompositions

It is quite common that, especially when dealing with Laplace or Z transform, one wants to apply the inverse transform to a rational function

$$\frac{P(s)}{Q(s)}.$$

If none of the standard rules apply directly, the standard approach is to first of all perform polynomial division if the degree of P is greater than or equal to the degree of Q. After this step it is usually the best approach to make a partial fraction decomposition.

Suppose now that $\deg P < \deg Q$. We know that the polynomial Q can be factored (in $\mathbb{R}$) into linear factors, $(s-a)$, and quadratic factors $\left((s-a)^2+b^2\right)$. Remember that a partial fraction decomposition is of the form

$$\frac{P(s)}{Q(s)} = \frac{p_1}{q_1} + \cdots + \frac{p_M}{q_M},$$

where $p_1, \ldots, p_M$ are constants or linear polynomials, and $q_1, \ldots, q_M$ consist of the linear and quadratic factors of Q (with *all* multiplicities). The following two general rules apply:

- A linear factor $(s-a)$ of multiplicity n contributes with

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}.$$

- A quadratic factor $\left((s-a)^2+b^2\right)$ of multiplicity n contributes with
$$\frac{A_1s+B_1}{((s-a)^2+b^2)}+\frac{A_2s+B_2}{((s-a)^2+b^2)^2}+\cdots+\frac{A_ns+B_n}{((s-a)^2+b^2)^n}.$$

The coefficients of the polynomials p_j are usually computed by putting the right hand side on a common denominator and comparing the resulting coefficients with $P(s)$.

Example 1.1. We consider the rational function
$$\frac{P(s)}{Q(s)}=\frac{3s^2+1}{s(s^2+1)(s-1)^2}.$$
The factors of Q are the linear factors s, $(s-1)$ of multiplicity 2 and the quadratic factor (s^2+1). Hence the partial fraction decomposition is
$$\frac{P(s)}{Q(s)}=\frac{3s^2+1}{s(s^2+1)(s-1)^2}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{Ds+E}{s^2+1},$$
and if we put the right hand side on a common denominator we get
$$\frac{3s^2+1}{s(s^2+1)(s-1)^2}=\frac{A(s-1)^2\left(s^2+1\right)+Bs(s-1)(s^2+1)+Cs(s^2+1)+(Ds+E)s(s-1)^2}{s(s^2+1)(s-1)^2},$$
and hence
$$3s^2+1=A(s-1)^2\left(s^2+1\right)+Bs(s-1)(s^2+1)+Cs(s^2+1)+(Ds+E)s(s-1)^2.$$
We can solve for A and C immediately: if we set $s=1$ we see that
$$3+1=A\cdot 0+B\cdot 0+2C+D\cdot 0+E\cdot 0=2C,$$
hence $C=2$, and if we set $s=0$ we see that $1=A$. Hence we must have
$$\begin{aligned}3s^2+1 &= (s-1)^2\left(s^2+1\right)+Bs(s-1)(s^2+1)+2s(s^2+1)+(Ds+E)s(s-1)^2\\ &= \left(s^2+1-2s\right)(s^2+1)+B(s^2-s)(s^2+1)+2s^3+2s+(Ds^2+Es)\left(s^2-2s+1\right)\\ &= s^4-2s^3+2s^2-2s+1+B(s^4-s^3+s^2-s)+2s^3+2s+Ds^4-2Ds^3+Es^3-2Es^2\\ &\quad +Ds^2+Es\\ &= s^4(1+B+D)+s^3(-2-B+2-2D+E)+s^2(2+B-2E+D)+s(-2-B+2+E)+1.\end{aligned}$$
And we get the following equations for the coefficients
$$\begin{aligned}1 &= 1,\\ E-B &= 0,\\ B+D-2E+2 &= 3,\\ E-2D-B &= 0,\\ B+D+1 &= 0,\end{aligned}$$
and using standard linear algebra we see that $B=E=-1$, and $D=0$. Hence the partial fraction decomposition becomes
$$\frac{P(s)}{Q(s)}=\frac{3s^2+1}{s(s^2+1)(s-1)^2}=\frac{1}{s}-\frac{1}{s-1}+\frac{2}{(s-1)^2}-\frac{1}{s^2+1}.$$
This can (and should) now also be verified by multiplying together all factors of the right hand side again.