CHAPTER 8

Integral Equations

8.1. Introduction

Integral equations appears in most applied areas and are as important as differential equations. In fact, as we will see, many problems can be formulated (equivalently) as either a differential or an integral equation.

Example 8.1. Examples of integral equations are:

(a) $$y(x) = x - \int_0^x (x-t)y(t)dt.$$

(b) $y(x) = f(x) + \lambda \int_0^x k(x-t)y(t)dt$, where $f(x)$ and $k(x)$ are specified functions.

(c) $y(x) = \lambda \int_0^1 k(x,t)y(t)dt$, where

$$k(x,t) = \begin{cases} x(1-t), & 0 \le x \le t, \\ t(1-x), & t \le x \le 1. \end{cases}$$

(d) $$y(x) = \lambda \int_0^1 (1-3xt)\, y(t)dt.$$

(e) $$y(x) = f(x) + \lambda \int_0^1 (1-3xt)\, y(t)dt.$$

♢

A general integral equation for an unknown function $y(x)$ can be written as

$$f(x) = a(x)y(x) + \int_a^b k(x,t)y(t)dt,$$

where $f(x), a(x)$ and $k(x,t)$ are given functions (the function $f(x)$ corresponds to an external force). The function $k(x,t)$ is called the *kernel*. There are different types of integral equations. We can classify a given equation in the following three ways.

- The equation is said to be of the *First kind* if the unknown function only appears under the integral sign, i.e. if $a(x) \equiv 0$, and otherwise of the *Second kind.*
- The equation is said to be a *Fredholm equation* if the integration limits a and b are constants, and a *Volterra equation* if a and b are functions of x.
- The equation are said to be *homogeneous* if $f(x) \equiv 0$ otherwise *inhomogen*eous.

Example 8.2. A Fredholm equation (Ivar Fredholm):

$$\int_a^b k(x,t)y(t)dt + a(x)y(x) = f(x).$$

◊

Example 8.3. A Volterra equation (Vito Volterra):

$$\int_a^x k(x,t)y(t)dt + a(x)y(x) = f(x).$$

◊

Example 8.4. The storekeeper's control problem.
To use the storage space optimally a storekeeper want to keep the stores stock of goods constant. It can be shown that to manage this there is actually an integral equation that has to be solved. Assume that we have the following definitions:

$$\begin{aligned} a &= \text{number of products in stock at time } t = 0, \\ k(t) &= \text{remainder of products in stock (in percent) at the time } t, \\ u(t) &= \text{the velocity (products/time unit) with which new products are purchased,} \\ u(\tau)\Delta\tau &= \text{the amount of purchased products during the time interval } \Delta\tau. \end{aligned}$$

The total amount of products at in stock at the time t is then given by:

$$ak(t) + \int_0^t k(t-\tau)u(\tau)d\tau,$$

and the amount of products in stock is constant if, for some constant c_0, we have

$$ak(t) + \int_0^t k(t-\tau)u(\tau)d\tau = c_0.$$

To find out how fast we need to purchase new products (i.e. $u(t)$) to keep the stock constant we thus need to solve the above Volterra equation of the first kind.

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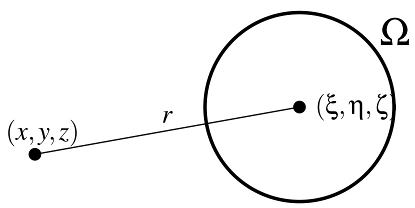
Example 8.5. (Potential)
Let $V(x,y,z)$ be the potential in the point (x,y,z) coming from a mass distribution $\rho(\xi,\eta,\zeta)$ in Ω (see Fig. 8.1.1). Then

$$V(x,y,z) = -G\int\int\int_\Omega \frac{\rho(\xi,\eta,\zeta)}{r}d\xi d\eta d\zeta.$$

The inverse problem, to determine ρ from a given potential V, gives rise to an integrated equation. Furthermore ρ and V are related via Poisson's equation

$$\nabla^2 V = 4\pi G\rho.$$

FIGURE 8.1.1. A potential from a mass distribution



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8.2. Integral Equations of Convolution Type

We will now consider integral equations of the following type:

$$y(x) = f(x) + \int_0^x k(x-t)y(t)dt = f(x) + k \star y(x),$$

where $k \star y(x)$ is the *convolution product* of k and y (see p. 45). The most important technique when working with convolutions is the Laplace transform (see sec. 6.2).

Example 8.6. Solve the equation

$$y(x) = x - \int_0^x (x-t)y(t)dt.$$

Solution: The equation is of convolution type with $f(x) = x$ and $k(x) = x$. We observe that $\mathcal{L}(x) = \frac{1}{s^2}$ and Laplace transforming the equation gives us

$$\begin{aligned} \mathcal{L}[y] &= \frac{1}{s^2} - \mathcal{L}[x \star y] = \frac{1}{s^2} - \mathcal{L}[x]\mathcal{L}[y] = \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}[y], \text{ i.e.} \\ \mathcal{L}[y] &= \frac{1}{1+s^2}, \end{aligned}$$

and thus $y(x) = \mathcal{L}^{-1}\left[\frac{1}{1+s^2}\right] = \sin x$.

Answer: $y(x) = \sin x$.

$\diamondsuit$

Example 8.7. Solve the equation

$$y(x) = f(x) + \lambda \int_0^x k(x-t)y(t)dt,$$

where $f(x)$ and $k(x)$ are fixed, given functions.

Solution: The equation is of convolution type, and applying the Laplace transform yields

$$\begin{aligned} \mathcal{L}[y] &= \mathcal{L}[f] + \lambda\mathcal{L}[k]\mathcal{L}[y], \text{ i.e.} \\ \mathcal{L}[y] &= \frac{\mathcal{L}[f]}{1-\lambda\mathcal{L}[k]}. \end{aligned}$$

Answer: $y(x) = \mathcal{L}^{-1}\left[\frac{\mathcal{L}[f]}{1-\lambda\mathcal{L}[k]}\right]$.

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8.3. The Connection Between Differential and Integral Equations (First-Order)

Example 8.8. Consider the differential equation (initial value problem)

$$\begin{cases} y'(x) &= f(x,y), \\ y(x_0) &= y_0. \end{cases} \tag{8.3.1}$$

By integrating from x_0 to x we obtain

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} f(t,y(t))dt,$$

i.e.

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt. \tag{8.3.2}$$

On the other hand, if (8.3.2) holds we see that $y(x_0) = y_0$, and

$$y'(x) = f(x,y(x)),$$

which implies that (8.3.1) holds! Thus the problems (8.3.1) and (8.3.2) are equivalent.

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In fact, it is possible to formulate many initial and boundary value problems as integral equations and vice versa. In general we have:

$$\left.\begin{array}{c}\text{Initial value problem}\\ \text{Dynamical system}\end{array}\right\} \Rightarrow \text{The Volterra equation,}$$

$$\text{Boundary value problem} \Rightarrow \text{The Fredholm equation.}$$

Picard's method (Emile Picard)

Problem: Solve the initial value problem

$$\begin{cases} y' &= f(x,y), \\ y(x_0) &= A. \end{cases}$$

Or equivalently, solve the integral equation :

$$y(x) = A + \int_{x_0}^{x} f(t,y(t))dt.$$

We will solve this integral equation by constructing a sequence of successive approximations to $y(x)$.

First choose an initial approximation, $y_0(x)$ (it is common to use $y_0(x) = y(x_0)$), then define the sequence: $y_1(x), y_2(x), \ldots, y_n(x)$ by

$$\begin{aligned} y_1(x) &= A + \int_{x_0}^{x} f(t,y_0(t))dt, \\ y_2(x) &= A + \int_{x_0}^{x} f(t,y_1(t))dt, \\ &\vdots \\ y_n(x) &= A + \int_{x_0}^{x} f(t,y_{n-1}(t))dt. \end{aligned}$$

Our hope is now that

$$y(x) \approx y_n(x).$$

By a famous theorem (Picard's theorem) we know that under certain conditions on $f(x,y)$ we have

$$y(x) = \lim_{n\to\infty} y_n(x).$$

Example 8.9. Solve the equation

$$\begin{cases} y'(x) &= 2x(1+y), \\ y(0) &= 0. \end{cases}$$

Solution: (With Picard's method) We have the integral equation

$$y(x) = \int_0^x 2t(1+y(t))dt,$$

and as the initial approximation we take $y_0(x) \equiv 0$. We then get

$$\begin{aligned} y_1(x) &= \int_0^x 2t(1+y_0(t))dt = \int_0^x 2t(1+0)dt = \int_0^x 2t dt = x^2, \\ y_2(x) &= \int_0^x 2t(1+y_1(t))dt = \int_0^x 2t(1+t^2)dt = \int_0^x 2t+2t^3 dt = x^2 + \frac{1}{2}x^4, \\ y_3(x) &= \int_0^x 2t(1+t^2+\frac{1}{2}t^4)dt = x^2 + \frac{1}{2}x^4 + \frac{x^6}{6}, \\ \vdots \quad & \quad \vdots \\ y_n(x) &= x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots + \frac{x^{2n}}{n!}. \end{aligned}$$

We see that

$$\lim_{n\to\infty} y_n(x) = e^{x^2} - 1.$$

REMARK 22. Observe that $y(x) = e^{x^2} - 1$ is the exact solution to the equation. (Show this!)

REMARK 23. In case one can can guess a general formula for $y_n(x)$ that formula can often be verified by, for example, induction.

LEMMA 8.1. *If $f(x)$ is continuous for $x \geq a$ then:*

$$\int_a^x \int_a^s f(y)dyds = \int_a^x f(y)(x-y)dy.$$

PROOF. Let $F(s) = \int_a^s f(y)dy$. Then we see that:

$$\begin{aligned} \int_a^x \int_a^s f(y)dyds &= \int_a^x F(s)ds = \int_a^x 1\cdot F(s)ds \\ \{\text{integration by parts}\} &= [sF(s)]_a^x - \int_a^x sF'(s)ds \\ &= xF(x) - aF(a) - \int_a^x sf(s)ds \\ &= x\int_a^x f(y)dy - 0 - \int_a^s yf(y)dy \\ &= \int_a^s f(y)(x-y)dy. \end{aligned}$$

□

8.4. The Connection Between Differential and Integral Equations (Second-Order)

Example 8.10. Assume that we want to solve the initial value problem

$$\begin{cases} u''(x)+u(x)q(x) & = f(x),\, x>a, \\ u(a)=u_0, & u'(a)=u_1. \end{cases} \tag{8.4.1}$$

We integrate the equation from a to x and get

$$u'(x)-u_1=\int_a^x [f(y)-q(y)u(y)]\,dy,$$

and another integration yields

$$\int_a^x u'(s)ds=\int_a^x u_1 ds+\int_a^x\int_a^s [f(y)-q(y)u(y)]\,dyds.$$

By Lemma 8.1 we get

$$u(x)-u_0=u_1(x-a)+\int_a^x [f(y)-q(y)u(y)]\,(x-y)dy,$$

which we can write as

$$\begin{aligned} u(x) &= u_0+u_1(x-a)+\int_a^x f(y)(x-y)dy+\int_a^x q(y)(y-x)u(y)dy \\ &= F(x)+\int_a^x k(x,y)u(y)dy, \end{aligned}$$

where

$$\begin{aligned} F(x) &= u_0+u_1(x-a)+\int_a^x f(y)(x-y)dy, \text{ and} \\ k(x,y) &= q(y)(y-x). \end{aligned}$$

This implies that (8.4.1) can be written as **Volterra equation**:

$$u(x)=F(x)+\int_a^x k(x,y)u(y)dy.$$

REMARK 24. Example 8.10 shows how an initial value problem can be transformed to an integral equation. In example 8.12 below we will show that an integral equation can be transformed to a differential equation, but first we need a lemma.

LEMMA 8.2. *(Leibniz's formula)*

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} u(x,t)dx\right)=\int_{a(t)}^{b(t)} u_t'(x,t)dx+u(b(t),t)b'(t)-u(a(t),t)a'(t).$$

PROOF. Let

$$G(t,a,b)=\int_a^b u(x,t)dx,$$

where

$$\begin{cases} a & = a(t), \\ b & = b(t). \end{cases}$$

The chain rule now gives

$$\begin{aligned}\frac{d}{dt}G &= G'_t(t,a,b)+G'_a(t,a,b)a'(t)+G'_b(t,a,b)b'(t)\\ &= \int_a^b u'_t(x,t)dx-u(a(t),t)a'(t)+u(b(t),t)b'(t).\end{aligned}$$

□

Example 8.11. Let

$$F(t)=\int_{\sqrt{t}}^{t^2}\sin(xt)dx.$$

Then

$$F'(t)=\int_{\sqrt{t}}^{t^2}\cos(xt)xdx+\sin t^3\cdot 2t-\sin t^{\frac{3}{2}}\cdot\frac{1}{2\sqrt{t}}.$$

♢

Example 8.12. Consider the equation

$$y(x)=\lambda\int_0^1 k(x,t)y(t)dt, \qquad (*)$$

where

$$k(x,t)=\begin{cases}x(1-t), & x\le t\le 1,\\ t(1-x), & 0\le t\le x.\end{cases}$$

I.e. we have

$$y(x)=\lambda\int_0^x t(1-x)y(t)dt+\lambda\int_x^1 x(1-t)y(t)dt.$$

If we differentiate $y(x)$ we get (using Leibniz's formula)

$$\begin{aligned}y'(x) &= \lambda\int_0^x -ty(t)dt+\lambda x(1-x)y(x)+\lambda\int_x^1(1-t)y(t)dt-\lambda x(1-x)y(t)\\ &= \lambda\int_0^x -ty(t)dt+\lambda\int_x^1(1-t)y(t)dt,\end{aligned}$$

and one further differentiation gives us

$$y''(x)=-\lambda xy(x)-\lambda(1-x)y(x)=-\lambda y(x).$$

Furthermore we see that $y(0)=y(1)=0$. Thus the integral equation (*) is equivalent to the boundary value problem

$$\begin{cases}y''(x)+\lambda y(x) &=0\\ y(0)=y(1) &=0.\end{cases}$$

8.5. A General Technique to Solve a Fredholm Integral Equation of the Second Kind

We consider the equation:

$$y(x) = f(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi. \tag{8.5.1}$$

Assume that the kernel $k(x,\xi)$ is *separable*, which means that it can be written as

$$k(x,\xi) = \sum_{j=1}^{n} \alpha_j(x) \beta_j(\xi).$$

If we insert this into (8.5.1) we get

$$\begin{aligned} y(x) &= f(x) + \lambda \sum_{j=1}^{n} \alpha_j(x) \int_a^b \beta_j(\xi) y(\xi) d\xi \\ &= f(x) + \lambda \sum_{j=1}^{n} c_j \alpha_j(x). \end{aligned} \tag{8.5.2}$$

Observe that $y(x)$ as in (8.5.2) gives us a solution to (8.5.1) as soon as we know the coefficients c_j. How can we find c_j?

Multiplying (8.5.2) with $\beta_i(x)$ and integrating gives us

$$\int_a^b y(x)\beta_i(x)dx = \int_a^b f(x)\beta_i(x)dx + \lambda \sum_{j=1}^{n} c_j \int_a^b \alpha_j(x)\beta_i(x)dx,$$

or equivalently

$$c_i = f_i + \lambda \sum_{j=1}^{n} c_j a_{ij}.$$

Thus we have a linear system with n unknown variables: $c_1, \dots, c_n$, and n equations $c_i = f_i + \lambda \sum_{j=1}^{n} c_j a_{ij}$, $1 \le i \le n$. In matrix form we can write this as

$$(I - \lambda A)\vec{c} = \vec{f},$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \text{ and } \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Some well-known facts from linear algebra

Suppose that we have a linear system of equations

$$B\vec{x} = \vec{b}. \tag{*}$$

Depending on whether the right hand side $\vec{b}$ is the zero vector or not we get the following alternatives.

1. If $\vec{b} = \vec{0}$ then:
   - a) $\det B \neq 0 \Rightarrow \vec{x} = \vec{0}$,
   - b) $\det B = 0 \Rightarrow$ (*) has an infinite number of solutions $\vec{x}$.
2. If $\vec{b} \neq 0$ then:
   - c) $\det B \neq 0 \Rightarrow$ (*) has a unique solution $\vec{x}$,
   - d) $\det B = 0 \Rightarrow$ (*) has no solution *or* an infinite number of solutions.

The famous *Fredholm Alternative Theorem* is simply a reformulation of the fact stated above to the setting of a Fredholm equation.

Example 8.13. Consider the equation

(*) $$y(x) = \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi.$$

Here we have

$$k(x,\xi) = 1 - 3x\xi = \alpha_1(x)\beta_1(\xi) + \alpha_2(x)\beta_2(\xi),$$

i.e.

$$\begin{cases} \alpha_1(x) = 1, & \alpha_2(x) = -3x, \\ \beta_1(\xi) = 1, & \beta_2(\xi) = \xi. \end{cases}$$

We thus get

$$A = \begin{pmatrix} \int_0^1 \beta_1(x)\alpha_1(x)dx & \int_0^1 \beta_1(x)\alpha_2(x)dx \\ \int_0^1 \beta_2(x)\alpha_1(x)dx & \int_0^1 \beta_2(x)\alpha_2(x)dx \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{pmatrix},$$

and

$$\begin{aligned} \det(I - \lambda A) &= \det \begin{pmatrix} 1-\lambda & \lambda\frac{3}{2} \\ -\lambda\frac{1}{2} & 1+\lambda \end{pmatrix} = 1 - \frac{\lambda^2}{4} = 0 \\ &\Leftrightarrow \\ \lambda &= \pm 2. \end{aligned}$$

The Fredholm Alternative Theorem tells us that we have the following alternatives:

$\lambda \neq \pm 2$ then (*) has only the trivial solution $y(x) = 0$, and

$\lambda = 2$ then the system $(I - \lambda A)\vec{c} = \vec{0}$ looks like

$$\begin{cases} -c_1 + 3c_2 &= 0, \\ -c_1 + 3c_2 &= 0, \end{cases}$$

which has an infinite number of solutions: $c_2 = a$ and $c_3 = 3a$, for any constant a. From (8.5.2) we see that the solutions $y(x)$ are

$$\begin{aligned} y(x) &= 0 + 2(3a \cdot 1 + a(-3x)) = 6a(1-x) \\ &= b(1-x). \end{aligned}$$

We conclude that every function $y(x) = b(1-x)$ is a solution of (*).

$\lambda = -2$ Then the system $(I - \lambda A)\vec{c} = \vec{0}$ looks like

$$\begin{cases} 3c_1 - 3c_2 &= 0, \\ c_1 - c_2 &= 0, \end{cases}$$

which has an infinite number of solutions $c_1 = c_2 = a$ for any constant a. From (8.5.2) we once again see that the solutions $y(x)$ are

$$\begin{aligned} y(x) &= 0 - 2(a \cdot 1 + a(-3x)) = -2a(1-3x) \\ &= b(1-3x), \end{aligned}$$

and we see that every function $y(x)$ of the form $y(x) = b(1-3x)$ is a solution of (*).

As always when solving a differential or integral equation one should test the solutions by inserting them into the equation in question. If we insert $y(x) = 1 - x$ and $y(x) = 1 - 3x$ in (*) we can confirm that they are indeed solutions corresponding to $\lambda = 2$ and -2 respectively.

$\diamondsuit$

Example 8.14. Consider the equation

(*) $$y(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)y(\xi)d\xi.$$

Note that the basis functions α_j and β_j and hence the matrix A is the same as in the previous example, and hence $\det(I - \lambda A) = 0 \Leftrightarrow \lambda = \pm 2$. The Fredholm Alternative Theorem gives us the following possibilities:

1° $\int_0^1 f(x) \cdot 1 dx \neq 0$ or $\int_0^1 f(x) \cdot x dx \neq 0$ and $\lambda \neq \pm 2$. Then $(*)$ has a unique solution

$$y(x) = f(x) + \lambda \sum_{i=1}^{2} c_i \alpha_i(x) = f(x) + \lambda c_1 - 3\lambda c_2 x,$$

where c_1 and c_2 is (the unique) solution of the system

$$\begin{cases} (1-\lambda)c_1 + \frac{3}{2}\lambda c_2 & = \int_0^1 f(x)dx, \\ -\frac{1}{2}\lambda c_1 + (1+\lambda)c_2 & = \int_0^1 xf(x)dx. \end{cases}$$

2° $\int_0^1 f(x) \cdot 1 dx \neq 0$ or $\int_0^1 f(x) \cdot x dx \neq 0$ and $\lambda = -2$. We get the system

$$\begin{cases} 3c_1 - 3c_2 & = \int_0^1 f(x)dx, \\ c_1 - c_2 & = \int_0^1 xf(x)dx. \end{cases}$$

Since the left hand side of the topmost equation is a multiple of the left hand side of the bottom equation there are *no solutions* if $\int_0^1 xf(x)dx \neq 3\int_0^1 f(x)dx$, and there are an *infinite number of solutions* if $\int_0^1 xf(x)dx = 3\int_0^1 f(x)dx$. Let $3c_2 = a$, then $3c_1 = a + \int_0^1 f(x)dx$, which gives the solutions

$$\begin{aligned} y(x) &= f(x) - 2\left[c_1\alpha_1(x) + c_2\alpha_2(x)\right] \\ &= f(x) - 2\left[\left(\frac{a}{3} + \frac{1}{3}\int_0^1 f(x)dx\right) + \frac{a}{3}(-3x)\right] \\ &= f(x) - \frac{2}{3}\int_0^1 f(x)dx - a\left(\frac{2}{3} - 2x\right). \end{aligned}$$

3° $\int_0^1 f(x) \cdot 1 dx \neq 0$ or $\int_0^1 f(x) \cdot x dx \neq 0$ and $\lambda = 2$. We get the system

$$\begin{cases} -c_1 + 3c_2 & = \int_0^1 f(x)dx, \\ -c_1 + 3c_2 & = \int_0^1 xf(x)dx. \end{cases}$$

The left hand sides are identical so there are *no solutions* if $\int_0^1 xf(x)dx \neq \int_0^1 f(x)dx$, otherwise we have an *infinite number of solutions*. Let $c_2 = a$, $c_1 = 3a - \int_0^1 f(x)dx$, then we get the solution

$$\begin{aligned} y(x) &= f(x) + 2\left[3a - \int_0^1 f(x)dx + a(-3x)\right] \\ &= f(x) - 2\int_0^1 f(x)dx + 6a(1-x). \end{aligned}$$

4° $\int_0^1 xf(x)dx = \int_0^1 f(x)dx = 0$, $\lambda \neq \pm 2$. Then $y(x) = f(x)$ is the unique solution.

5° $\int_0^1 xf(x)dx = \int_0^1 f(x)dx = 0$, $\lambda = -2$. We get the system

$$\begin{cases} 3c_1 - 3c_2 &= 0, \\ c_1 - c_2 &= 0, \end{cases} \Leftrightarrow \quad c_1 = c_2 = a,$$

for an arbitrary constant a.We thus get an *infinite number of solutions* of the form

$$\begin{aligned} y(x) &= f(x) - 2\left[a \cdot 1 + a(-3x)\right] \\ &= f(x) - 2a(1-3x). \end{aligned}$$

6° $\int_0^1 xf(x)dx = \int_0^1 f(x)dx = 0$, $\lambda = 2$. We get the system

$$\begin{cases} -c_1 + 3c_2 &= 0, \\ c_1 + 3c_2 &= 0, \end{cases} \Leftrightarrow \quad \begin{cases} c_2 &= a, \\ c_1 &= 3a, \end{cases}$$

for an arbitrary constant a. We thus get an *infinite number of solutions* of the form

$$\begin{aligned} y(x) &= f(x) + 2\left[3a \cdot 1 + a(-3x)\right] \\ &= f(x) + 6a(1-x). \end{aligned}$$

8.6. Integral Equations with Symmetrical Kernels

Consider the equation

(*) $$y(x) = \lambda \int_a^b k(x,\xi) y(\xi) d\xi,$$

where

$$k(x,\xi) = k(\xi,x)$$

is real and continuous. We will now see how we can adapt the theory from the previous sections to the case when $k(x,\xi)$ is not separable but instead is symmetrical, i.e. $k(x,\xi) = k(\xi,x)$. If λ and $y(x)$ satisfy (*) we say that λ is an *eigenvalue* and $y(x)$ is the corresponding *eigenfunction*. We have the following theorem.

THEOREM 8.3. *The following holds for eigenvalues and eigenfunctions of (*):*

(i) if λ_m and λ_n are eigenvalues with corresponding eigenfunctions $y_m(x)$ and $y_n(x)$ then:

$$\lambda_n \neq \lambda_m \Rightarrow \int_a^b y_m(x) y_n(x) dx = 0.$$

I.e. eigenfunctions corresponding to different eigenvalues are orthogonal ($y_m(x) \perp y_n(x)$).

(ii) The eigenvalues λ are real.

(iii) If the kernel k is not separable then there are infinitely many eigenvalues:

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots,$$

with $0 < |\lambda_1| \le |\lambda_2| \le \cdots$ and $\lim_{n\to\infty} |\lambda_n| = \infty$.

(iv) To every eigenvalue corresponds at most a finite number of linearly independent eigenfunctions.

PROOF. (i). We have

$$\begin{aligned} y_m(x) &= \lambda_m \int_a^b k(x,\xi) y_m(\xi) d\xi, \text{ and} \\ y_n(x) &= \lambda_n \int_a^b k(x,\xi) y_n(\xi) d\xi, \end{aligned}$$

which gives

$$\begin{aligned} \int_a^b y_m(x) y_n(x) dx &= \lambda_m \int_a^b y_n(x) \int_a^b k(x,\xi) y_m(\xi) d\xi dx \\ &= \lambda_m \int_a^b \left(\int_a^b y_n(x) k(k,\xi) dx \right) y_m(\xi) d\xi \\ [k(x,\xi) = k(\xi,x)] &= \lambda_m \int_a^b \left(\int_a^b k(\xi,x) y_n(x) dx \right) y_m(\xi) d\xi \\ &= \lambda_m \int_a^b \left(\frac{1}{\lambda_n} y_n(\xi) \right) y_m(\xi) d\xi \\ &= \frac{\lambda_m}{\lambda_n} \int_a^b y_m(\xi) y_n(\xi) d\xi. \end{aligned}$$

We conclude that

$$\left(1 - \frac{\lambda_m}{\lambda_n}\right) \int_a^b y_m(x) y_n(x) dx = 0,$$

and if $\lambda_m \neq \lambda_n$ then we must have $\int_a^b y_m(x) y_n(x) dx = 0$. □

Example 8.15. Solve the equation

$$y(x) = \lambda \int_0^1 k(x,\xi) y(\xi) d\xi,$$

where

$$k(x,\xi) = \begin{cases} x(1-\xi), & x \le t \le 1, \\ \xi(1-x), & 0 \le \xi \le x. \end{cases}$$

From Example 8.12 we know that the integral equation is equivalent to

$$\begin{cases} y''(x) + \lambda y(x) &= 0, \\ y(0) = y(1) &= 0. \end{cases}$$

If $\lambda > 0$ we have the solutions $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$, $y(0) = 0 \Rightarrow c_1 = 0$ and $y(1) = 0$ $\Rightarrow c_2 \sin\sqrt{\lambda} = 0$, hence either $c_2 = 0$ (which only gives the trivial solution $y \equiv 0$) or $\sqrt{\lambda} = n\pi$ for some integer n, i.e. $\lambda = n^2\pi^2$. Thus, the eigenvalues are

$$\lambda_n = n^2\pi^2,$$

and the corresponding eigenfunctions are

$$y_n(x) = \sin(n\pi x).$$

Observe that if $m \neq n$ it is well-known that

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = 0.$$

$\diamondsuit$

8.7. Hilbert-Schmidt Theory to Solve a Fredholm Equation

We will now describe a method for solving a Fredholm Equation of the type:

$$(*) \qquad y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt.$$

LEMMA 8.4. *(Hilbert-Schmidt's Lemma) Assume that there is a continuous function $g(x)$ such that*

$$F(x) = \int_a^b k(x,t)g(t)dt,$$

where k is symmetrical (i.e. $k(x,t) = k(t,x)$). Then $F(x)$ can be expanded in a Fourier series as

$$F(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

where $y_n(x)$ are the normalized eigenfunctions to the equation

$$y(x) = \lambda \int_a^b k(x,t)y(t)dt.$$

(Cf. Thm. 8.3.)

THEOREM 8.5. *(The Hilbert-Schmidt Theorem) Assume that λ **is not** an eigenvalue of (*) and that $y(x)$ is a solution to (*). Then*

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} y_n(x),$$

where λ_n and $y_n(x)$ are eigenvalues and eigenfunctions to the corresponding homogeneous equation (i.e. (*) with $f \equiv 0$) and $f_n = \int_a^b f(x)y_n(x)dx$.

PROOF. From (*) we see immediately that

$$y(x) - f(x) = \lambda \int_a^b k(x,\xi)y(\xi)d\xi,$$

and according to H-S Lemma (8.4) we can expand $y(x) - f(x)$ in a Fourier series:

$$y(x) - f(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

where

$$c_n = \int_a^b (y(x) - f(x))\, y_n(x)dx = \int_a^b y(x)y_n(x)dx - f_n.$$

Hence

$$\begin{aligned}
\int_a^b y(x)y_n(x)dx &= f_n + \int_a^b (y(x)-f(x))\,y_n(x)dx \\
&= f_n + \lambda \int_a^b \left(\int_a^b k(x,\xi)y(\xi)d\xi\right) y_n(x)dx \\
\{k(x,\xi)=k(\xi,x)\} \quad &= f_n + \lambda \int_a^b \left(\int_a^b k(\xi,x)y_n(x)dx\right) y(\xi)d\xi \\
&= f_n + \frac{\lambda}{\lambda_n}\int_a^b y_n(\xi)y(\xi)d\xi.
\end{aligned}$$

Thus

$$\int_a^b y(x)y_n(x)dx = \frac{f_n}{1-\frac{\lambda}{\lambda_n}} = \frac{\lambda_n f_n}{\lambda_n-\lambda},$$

and we conclude that

$$c_n = \frac{\lambda_n f_n}{\lambda_n-\lambda} - f_n = \frac{\lambda f_n}{\lambda_n-\lambda},$$

i.e. we can write $y(x)$ as

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n-\lambda} y_n(x).$$

□

Example 8.16. Solve the equation

$$y(x) = x + \lambda \int_0^1 k(x,\xi)y(\xi)d\xi,$$

where $\lambda \neq n^2\pi^2$, $n = 1,2,\ldots$, and

$$k(x,\xi) = \begin{cases} x(1-\xi), & x \le \xi \le 1, \\ \xi(1-x), & 0 \le \xi \le x. \end{cases}$$

Solution: From Example 8.15 we know that the normalized eigenfunctions to the homogeneous equation

$$y(x) = \lambda \int_0^1 k(x,\xi)y(x)d\xi$$

are

$$y_n(x) = \sqrt{2}\sin(n\pi x),$$

corresponding to the eigenvalues $\lambda_n = n^2\pi^2$, $n = 1,2,\ldots$. In addition we see that

$$f_n = \int_0^1 f(x)y_n(x)dx = \int_0^1 x\sqrt{2}\sin(n\pi x)\,dx = \frac{(-1)^{n+1}\sqrt{2}}{n\pi},$$

hence

$$y(x) = x + \frac{\sqrt{2}\lambda}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,(n^2\pi^2-\lambda)}\sin(n\pi x)\,,\ \lambda \neq n^2\pi^2.$$

◇

Finally we observe that by using practically the same ideas as before we can also prove the following theorem (cf. (5, pp. 246-247)).

THEOREM 8.6. *Let f and k be continuous functions and define the operator K acting on the function $y(x)$ by*

$$Ky(x) = \int_a^x k(x,\xi)y(\xi)d\xi,$$

and then define positive powers of K by

$$K^m y(x) = K(K^{m-1}y)(x), m = 2,3,\ldots.$$

Then the equation

$$y(x) = f(x) + \lambda \int_a^x k(x,\xi)y(\xi)d\xi$$

has the solution

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n K^n(f).$$

This type of series expansion is called a **Neumann series**.

Example 8.17. Solve the equation

$$y(x) = x + \lambda \int_0^x (x-\xi)\, y(\xi)d\xi.$$

Solution: (by Neumann series):

$$\begin{aligned}
K(x) &= \int_0^x (x-\xi)\xi d\xi = \frac{x^3}{3!} \\
K^2(x) &= \int_0^x (x-\xi)\,\frac{\xi^3}{3!}d\xi = \frac{x^5}{5!} \\
&\vdots \\
K^n(x) &= \int_0^x (x-\xi)\,\frac{\xi^{2n-1}}{(2n-1)!}d\xi = \frac{x^{2n+1}}{(2n+1)!},
\end{aligned}$$

hence

$$\begin{aligned}
y(x) &= x + \sum_{n=1}^{\infty} \lambda^n K^n(x) \\
&= x + \lambda\frac{x^3}{3!} + \lambda^2\frac{x^5}{5!} + \cdots + \lambda^n \frac{x^{2n+1}}{(2n+1)!} + \cdots.
\end{aligned}$$

Solution (by the Laplace transform): We observe that the operator

$$K = \int_0^x (x-\xi)\, y(\xi)d\xi$$

is a convolution of the function $y(x)$ with the identity function $x \mapsto x$, i.e. $K(x) = (t \mapsto t \star y)(x)$, which implies that $\mathcal{L}[K(x)] = \mathcal{L}[x]\mathcal{L}[y]$, and since $y(x) = x + \lambda K(x)$ we get

$$\begin{aligned}
\mathcal{L}(y) &= \mathcal{L}(x) + \lambda\mathcal{L}(x)\,\mathcal{L}(y) = \frac{1}{s^2} + \lambda\frac{1}{s^2}\mathcal{L}(y) \\
\mathcal{L}(y) &= \frac{1}{s^2-\lambda} = \frac{1}{2\sqrt{\lambda}}\left(\frac{1}{s-\sqrt{\lambda}} - \frac{1}{s+\sqrt{\lambda}}\right),
\end{aligned}$$

and by inverting the transform we get

$$y(x) = \frac{1}{2\sqrt{\lambda}}\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right).$$

Observe that we obtain the same solution independent of method. This is easiest seen by looking at the Taylor expansion of the second solution. More precisely we have

$$\begin{aligned} e^{-\sqrt{\lambda}x} &= 1-\sqrt{\lambda}x+\frac{1}{2}\left(\sqrt{\lambda}x\right)^2-\frac{1}{3!}\left(\sqrt{\lambda}x\right)^3+\cdots, \\ e^{\sqrt{\lambda}x} &= 1+\sqrt{\lambda}x+\frac{1}{2}\left(\sqrt{\lambda}x\right)^2+\frac{1}{3!}\left(\sqrt{\lambda}x\right)^3+\cdots, \end{aligned}$$

i.e.

$$\begin{aligned} y(x) &= \frac{1}{2\sqrt{\lambda}}\left(e^{\sqrt{\lambda}x}-e^{-\sqrt{\lambda}x}\right) \\ &= \frac{1}{2\sqrt{\lambda}}\left(2\sqrt{\lambda}x+\frac{2}{3!}\left(\sqrt{\lambda}x\right)^3+\frac{2}{5!}\left(\sqrt{\lambda}x\right)^5+\cdots\right) \\ &= x+\lambda\frac{x^3}{3!}+\lambda^2\frac{x^5}{5!}+\cdots. \end{aligned}$$

$\diamondsuit$

8.8. Exercises

8.1. [A] Rewrite the following second order initial values problem as an integral equation

$$\begin{cases} u''(x)+p(x)u'(x)+q(x)u(x) & = f(x), x>a, \\ u(a)=u_0, & u'(a)=u_1. \end{cases}$$

8.2. Consider the initial values problem

$$\begin{cases} u''(x)+\omega^2u(x) & = f(x), x>0, \\ u(a)=0, & u'(0)=1. \end{cases}$$

a) Rewrite this equation as an integral equation.

b) Use the Laplace transform to give the solution for a general $f(x)$ with Laplace transform $F(s)$.

c) Give the solution $u(x)$ for $f(x)=\sin at$ with $a\in\mathbb{R}$, $a\neq\omega$.

8.3. [A] Rewrite the initial values problem

$$\begin{aligned} y''(x)+\omega^2y &= 0, 0\le x\le 1, \\ y(0)=1, y'(0) &= 0 \end{aligned}$$

as an integral equation of Volterra type and give those solutions which also satisfy $y(1)=0$.

8.4. Rewrite the boundary values problem

$$\begin{aligned} y''(x)+\lambda p(x)y &= q(x), a\le x\le b, \\ y(a)=y(b) &= 0 \end{aligned}$$

as an integral equation of Fredholm type. (Hint: Use $y(b)=0$ to determine $y'(a)$.)

8.5. **[A]** Let $\alpha \geq 0$ and consider the probability that a randomly chosen integer between 1 and x has its largest prime factor $\leq x^{\alpha}$. As $x \to \infty$ this probability distribution tends to a limit distribution with the distribution function $F(\alpha)$, the so called Dickman function (note that $F(\alpha) = 1$ for $\alpha \geq 1$). The function $F(\alpha)$ is a solution of the following integral equation

$$F(\alpha) = \int_0^{\alpha} F\left(\frac{t}{1-t}\right)\frac{1}{t}dt,\ 0 \leq \alpha \leq 1.$$

Compute $F(\alpha)$ for $\frac{1}{2} \leq \alpha \leq 1$.

8.6.* Consider the Volterra equation

$$u(x) = x + \mu \int_0^x (x-y)\,u(y)dy.$$

a) Compute the first three non-zero terms in the Neumann series of the solution.

b) Give the solution of the equation (for example by using a) to guess a solution and then verify it).

8.7. **[A]** Solve the following integral equation:

$$x = \int_0^x e^{x-\xi}y(\xi)d\xi.$$

8.8. Use the Laplace transform to solve:

a) $$y(x) = f(x) + \lambda \int_0^x e^{x-\xi}y(\xi)d\xi.$$

b) $$y(x) = 1 + \int_0^x e^{x-\xi}y(\xi)d\xi$$

8.9. **[A]** Write a Neumann series for the solution of the integral equation

$$u(x) = f(x) + \lambda \int_0^1 u(t)dt,$$

and give the solution of the equation for $f(x) = e^x - \frac{e}{2} + \frac{1}{2}$ and $\lambda = \frac{1}{2}$.

8.10. Solve the following integral equations:

a) $$y(x) = x^2 + \int_0^1 (1-3x\xi)\,y(\xi)d\xi,$$

b) $$y(x) = x^2 + \lambda \int_0^1 (1-3x\xi)\,y(\xi)d\xi \text{ for all values of } \lambda.$$

8.11. **[A]** Solve the following integral equation

$$u(x) = f(x) + \lambda \int_0^{\pi} \sin(x)\sin(2y)\,u(x)dy$$

when

a) $f(x) = 0,$

b) $f(x) = \sin x,$

c) $f(x) = \sin 2x.$

8.12. Consider the equation

$$u(x) = f(x) + \lambda \int_0^x u(t)dt, \; 0 \leq x \leq 1.$$

a) Show that for $f(x) \equiv 0$ the equation has only the trivial solution in $C^2[0,1]$.
b) Give a function $f(x)$ such that the equation has a non-trivial solution for all values of λ and compute this solution.

8.13. **[A]** Let $a > 0$ and consider the integral equation

$$u(x) = 1 + \lambda \int_0^{x-a} \theta(x-y+a)(x-y)u(y)dy, \; x \geq a.$$

Use the Laplace transform to determine the eigenvalues and the corresponding eigenfunctions of this equation.

8.14.* The current in an LRC-circuit with $L = 3, R = 2, C = 0.2$ (SI-units) and where we apply a voltage at the time $t = 3$ satisfies the following integral equation

$$I(t) = 6\theta(t-1)(t-1) + 2t + 3 - \int_0^t (2 + 5(t-y)) I(y)dy.$$

Determine $I(t)$ using the Laplace transform.

8.15. **[A]** Consider (again) the salesman's control problem (Example 8.4). Assume that the number of products in stock at the time $t = 0$ is a and that the products are sold at a constant rate such that all products are sold out in T (time units). Now let $u(t)$ be the rate (products/time unit) with which we have to purchase new products in order to have a constant number of a products in stock.
a) Write the integral equation which is satisfied by $u(t)$.
b) Solve the equation from a) and find $u(t)$.
b) $u(t) = \frac{a}{T} e^{t/T}$.

8.16.*
a) Write the integral equation

(*) $$y(x) = \lambda \int_0^1 k(x,\xi) y(\xi) d\xi,$$

where

$$k(x,\xi) = \begin{cases} x(1-\xi), & x \leq \xi \leq 1, \\ \xi(1-x), & 0 \leq \xi \leq x, \end{cases}$$

as a boundary value problem.
b) Find the eigenvalues and the normalized eigenvectors to the problem in a).

Solve the equation

$$y(x) = f(x) + \lambda \int_0^1 k(x,\xi) y(\xi) d\xi,$$

where $k(x,\xi)$ is as in a) and $\lambda \neq n^2\pi^2$ for
c) $f(x) = \sin(\pi k x)$, $k \in \mathbb{Z}$, and
d) $f(x) = x^2$.

8.17. **[A]** Consider the Fredholm equation

$$u(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+t)\, u(t) dt.$$

Determine solutions for all values of λ and give sufficient conditions (if there are any) that $f(x)$ has to satisfy in order for solutions to exist.

8.18. Show that the equation

$$g(s) = \lambda \int_0^{\pi} (\sin s \sin 2t)\, g(t) dt$$

only has the trivial solution.

8.19. **[A]** Solve the integral equation

$$\sin s = \frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{u(t)}{t-s} dt,$$

where $\int^*$ means that we consider the principal value of the integral (since the integrand has a singularity at $t = s$). (Hint: Use the residue theorem on the integral $\int_{-\infty}^{\infty} \frac{e^{it}}{s-t} dt$.)

8.20. Give the Laplace transform of the non-trivial solution for the following integral equation

$$g(s) = \int_0^s \left(s^2 - t^2\right) g(t) dt.$$

Hint: Rewrite the kernel in convolution form and use the differentiation rule.