A Basic Course in Applied Mathematics (10 p)

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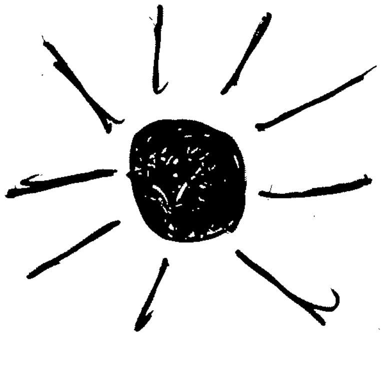
- 10 occasions, where three (*)-problems are pointed out each time.
- These (*) problems are important for the final examination which can be either
  - a) usual written examination

  or
  - b) oral examination mainly with questions connected to these (*) problems



1. THE PROGRAM OF APPLIED MATHEMATICS

TECHNICAL
OR
PHYSICAL PROBLEM

A FORMULATE A MATHEMATICAL MODEL OF THE SITUATION e.g. FORMULATE THE ACTUAL GOVERNING EQUATIONS.

B SOLVE THE EQUATIONS BY USING SOME ANALYTICAL AND/OR NUMERICA METHOD.

C GO BACK AND VERIFY THAT THE OBTAINED SOLUTION IS CONSISTENT WITH THE EXPERIMENTAL OBSERV-ATIONS.

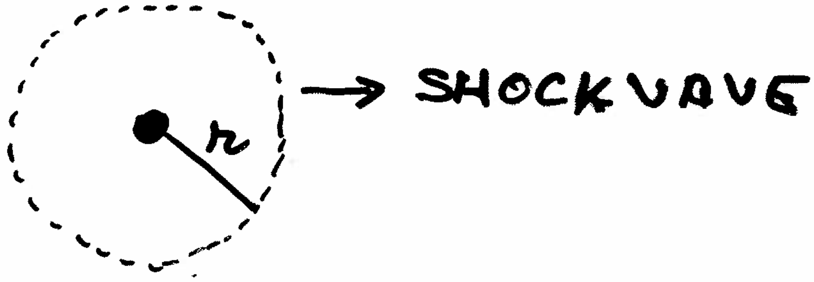
<u>REMARK</u>: IN STEP A WE CAN HAVE ADDITIONAL SUPPORT BY USING

a) DIMENSIONAL ANALYSIS

b) SCALING.

2. AN INTRODUCTORY EXAMPLE (TAYLOR ≈ 19

ATOMIC EXPLOSION



WE **ASSUME** THAT THERE IS A PHYSICA
LAW

$$(*) \quad g(t, r, \rho, e) = 0$$

WITH

		DIMENSION
t:	TIME	T
r:	LENGTH	L
e:	ENERGY	ML^2/T^2
ρ:	DENSITY	M/L^3

HOW CAN WE GET A **DIMENSIONLESS** VARIABLE FROM THESE QUANTITIES?

ANSWER: FIND α_1, α_2, α_3 and α_4 SUCH THAT

$$T^{\alpha_1} L^{\alpha_2} (ML^2/T^2)^{\alpha_3} (M/L^3)^{\alpha_4} = 1$$

$\Leftrightarrow$

$$T^{\alpha_1 - 2\alpha_3} L^{\alpha_2 + 2\alpha_3 - 3\alpha_4} M^{\alpha_3 + \alpha_4} = 1$$

$\Leftrightarrow$

$$\begin{cases} \alpha_1 - 2\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 - 3\alpha_4 = 0 \\ \alpha_3 + \alpha_4 = 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

SOLUTIONS: $\alpha_4 = u$, $\alpha_3 = -u$, $\alpha_2 = 5u$, $\alpha_1 = -2u$

i.e.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = u \begin{pmatrix} -2 \\ 5 \\ -1 \\ 1 \end{pmatrix}$$ (u is an arbitrary real number).

E.G. BY CHOOSING $u = 1$ WE GET

$\alpha_1 = -2$, $\alpha_2 = 5$, $\alpha_3 -1$ and $\alpha_4 = 1$

THEREFORE

$$\pi = t^{\alpha_1} r^{\alpha_2} e^{\alpha_3} \cdot \rho^{\alpha_4} = \frac{r^5 \rho}{t^2 e}$$

IS A DIMENSIONLESS VARIABLE

THEREFORE WE KNOW (FROM THE **PI-THEOREM**) THAT ⊛ CAN BE WRITTEN (EQUIVALENTLY) AS

$$f\Big(\overbrace{\frac{r^5 \rho}{t^2 e}}^{\pi}\Big) = 0$$

i.e.

$$\frac{r^5 \rho}{t^2 e} = C \qquad (C \text{ constant})$$

WE CONCLUDE

$$r = C_0 \left(\frac{e t^2}{\rho}\right)^{1/5}.$$

3. A GENERALIZATION OF THE SITUATION

WE CONSIDER A PHYSICAL LAW

$$(1) \quad f(q_1, q_2, \ldots, q_m) = 0$$

FUNDAMENTAL DIMENSIONS: $L_1, L_2, \ldots, L_n$.
$n < m$.

THE DIMENSION OF q_i, denoted $[q_i]$, is

$$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \quad L_n^{a_{ni}}$$

THE DIMENSION MATRIX is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & a_{nm} \end{pmatrix}$$

CHANGE OF UNITS:

$$\bar{L}_i = \lambda_i L_i, \quad i = 1, 2, \ldots n \quad (\lambda_i > 0)$$

IF q HAS DIMENSION

$$[q] = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_n^{\alpha_n}$$

THEN

$$\bar{q} = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_n^{\alpha_n} q$$

GIVES q IN THE NEW SYSTEM

4. A UNIT FREE PHYSICAL LAW

THE PHYSICAL LAW (1) IS **UNIT FREE** IF FOR ALL POSITIVE λ_i

$$f(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_m) = 0 \quad \text{IF AND ONLY IF}$$

$$f(q_1, q_2, \cdots, q_m) = 0.$$

EXAMPLE: THE PHYSICAL LAW

$$f(x,t,g) = x - \frac{1}{2} g t^2 = 0$$

IS UNIT FREE.

PROOF: $\bar{x} = \lambda_1 x$, $\bar{t} = \lambda_2 t$ GIVES THAT

$$\bar{g} = \frac{\lambda_1}{\lambda_2^2} g \qquad \left([g] = \frac{L}{T^2} \right)$$

THUS

$$f(\bar{x}, \bar{t}, \bar{g}) = \bar{x} - \frac{1}{2} \bar{g} \bar{t}^2 = \lambda_1 x - \frac{1}{2} \frac{\lambda_1}{\lambda_2^2} g \lambda_2^2 t^2$$

$$= \lambda_1 \left(x - \frac{1}{2} g t^2 \right) = \lambda_1 f(x,t,g)$$

i.e.

$$f(\bar{x}, \bar{t}, \bar{g}) = 0 \iff f(x,t,g) = 0.$$

REMARK: IF x IS GIVEN IN CM AND $\bar{x}$ IN M THEN $\lambda_1 = 1/100$

IF t IS GIVEN IN SECONDS AND $\bar{t}$ IN MIN. THEN $\lambda_2 = 1/60$.

IN THIS SITUATION $\bar{g}$ IS GIVEN IN M/MIN^2.

5. THE Pi THEOREM

THEOREM: LET

$$(*) \quad f(q_1, q_2, \dots, q_m) = 0$$

BE A UNIT FREE PHYSICAL LAW THAT RELATES THE DIMENSIONAL QUANTITIES $q_1, q_2, \dots q_m$. LET $L_1, L_2, \dots, L_n$ $(n<m)$ BE THE FUNDAMENTAL DIMENSIONS WITH

$$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \dots L_n^{a_{ni}}, \quad i = 1, 2, \dots, m$$

AND LET $r = \text{rank}\, A$, WHERE A IS THE DIMENSION MATRIX. THEN THERE EXISTS $m-r$ INDEPENDENT DIMENSIONLESS QUANTITIES $\pi_1, \pi_2, \dots \pi_{m-r}$ WHICH CAN BE FORMED FROM $q_1, q_2, \dots q_m$ AND $(*)$ IS EQUIVALENT TO AN EQUATION

$$(**) \quad F(\pi_1, \pi_2, \dots, \pi_{m-r}) = 0$$

EXPRESSED ONLY IN TERMS OF DIMENSIONLESS QUANTITIES.

EX

* $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

HAS RANK 3

* $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

HAS RANK 2.

* $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 1 & 4 & 7 & 4 \end{pmatrix}$$

HAS RANK 2

P1 $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 7 \end{vmatrix} = 0$ $\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 0 \\ 1 & 7 & 4 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 0 \\ 4 & 7 & 4 \end{vmatrix} = 0$ $\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 1 & 4 & 4 \end{vmatrix} = 0$

but $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

P2 $r_3 = r_1 + 2r_2$ but $r_2 \neq c\, r_1$

P3

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 1 & 4 & 7 & 4 \end{pmatrix} \underset{r_3 - r_1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{pmatrix} \sim$$

$$\sim \underset{r_3 - 2r_2}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

* $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}$$

HAS RANK 1.

* $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

HAS RANK 3

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \end{pmatrix}$$

HAS RANK 2

EXAMPLE: IN OUR INTRODUCTORY EXAMPLE WE HAD

$$q_1 = t$$
$$q_2 = r$$
$$q_3 = e$$
$$q_4 = \rho \qquad \boxed{m=4}$$

FUNDAMENTAL DIMENSIONS: T, L, M $\boxed{n=3}$

$$[q_1] = T = T^1 L^0 M^0$$
$$[q_2] = L = T^0 L^1 M^0$$
$$[q_3] = \frac{ML^2}{T^2} = T^{-2} L^2 M^1$$
$$[q_4] = \frac{M}{L^3} = T^0 L^{-3} M^1$$

DIMENSION MATRIX

$$A = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\text{RANK } A = 3 \qquad \boxed{r=3}$$

WE HAVE $m - r = 4 - 3 = 1$ DIMENSIONLESS VARIABLE.

Part of the Proof the π-theorem:

let π be the dimensionless quantity.

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$$

Fundamental dimensions: $L_1, \dots, L_n$

$$\pi = \left(L_1^{a_{11}} \cdots L_n^{a_{n1}}\right)^{\alpha_1} \left(L_1^{a_{12}} \cdots L_n^{a_{n2}}\right)^{\alpha_2} \cdots \left(L_1^{a_{1m}} \cdots L_n^{a_{nm}}\right)^{\alpha_m}$$

$[\pi] = 1 \Leftrightarrow$

$$\begin{cases} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1m}\alpha_m = 0 \\ \vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nm}\alpha_m = 0 \end{cases}$$

m unknown, n equations $m > n$ $\boxed{\text{rank} = n}$

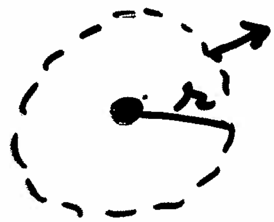
From Linear Algebra we know that there are $m-n$ independent solutions and each solution gives rise of a dimensionless variable.

6. ANOTHER MODEL EXAMPLE

$u = u(r,t)$

HEAT CONDUCTION:

$$f(t, r, u, e, k, c) = 0$$



t TIME $[t] = T$

r LENGTH $[r] = L$

u TEMPERATURE $[u] = \Theta$

e ENERGY $[e] = E$

k THERMAL DIFFUSIVITY $[k] = L^2/T$

c HEAT CAPACITY $[c] = E/\Theta L^3$ $\boxed{m=6}$

THE DIMENSION MATRIX IS $\boxed{n=4}$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad \boxed{r=4}$$

∴ WE HAVE $m - r = 6 - 4 = 2$ DIMENSIONLESS QUANTITIES.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

PUT $\alpha_6 = u$, $\alpha_5 = v$. THEN

$$\alpha_4 = -u,\quad \alpha_3 = u,\quad \alpha_2 = -2v + 3u,\quad \alpha_1 = v$$

i.e.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = u \begin{pmatrix} 0 \\ 3 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + v \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

IN THE BOOK THEY CHOOSE THE LINEARLY INDEPENDENT SOLUTIONS

1) $\alpha_1 = -1/2$, $\alpha_2 = 1$, $\alpha_3 = \alpha_4 = \alpha_6 = 0$, $\alpha_5 = -1/2$
   $(u = 0,\ v = -1/2)$

and

2) $\alpha_1 = 3/2$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = -1$, $\alpha_5 = 3/2$, $\alpha_6 = 1$
   $(u = 1,\ v = 3/2)$

WE GET THE TWO DIMENSIONLESS VARIABLES

$$\pi_1 = t^{-1/2} r^{1} u^{0} e^{0} k^{-1/2} c^{0} = r/\sqrt{kt}$$

AND

$$\pi_2 = t^{3/2} r^{0} u^{1} e^{-1} k^{3/2} c^{1} = \frac{uc}{e}(tk)^{3/2}$$

ANOTHER POSSIBILITY:

$$\pi_1 = r^3 u e^{-1} c = r^3 uc/e$$

$$\pi_2 = t r^{-2} k = tk/r^2$$

WE CONCLUDE THAT THERE IS AN (1
EQUIVALENT RELATION

$$F(\Pi_1, \Pi_2) = 0 \quad \text{i.e.} \quad \Pi_2 = g(\Pi_1)$$

BY THE Pi-THEOREM, THIS MEANS THAT

$$u = \frac{e}{c} \cdot \frac{1}{(tk)^{3/2}} g\left(\frac{r}{\sqrt{kt}}\right)$$

WITH THE OTHER POSSIBILITY WE GET

$$\frac{r^3 uc}{e} = g_0\left(\frac{tk}{r^2}\right)$$

$$\Leftrightarrow \quad u = \frac{e}{c} \cdot \frac{1}{r^3} g_0\left(\frac{tk}{r^2}\right) = \left[g_0(x) = g\left(\frac{1}{\sqrt{x}}\right) \cdot \left(\frac{1}{x}\right)^{3/2}\right.$$

$$= \frac{e}{c} \frac{1}{(tk)^{3/2}} \cdot g\left(\frac{r}{\sqrt{tk}}\right)$$

7. CHARACTERISTIC SCALES

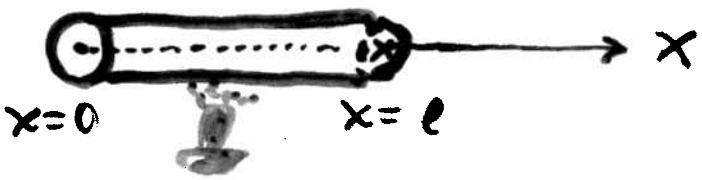
EXAMPLE: t = TIME

t_c = "CHARACTERISTIC TIME"

$$\bar{t} = \frac{t}{t_c} = \text{DIMENSIONLESS TIME}$$

REMARK: IN DIFFERENT PROBLEMS WE HAVE VERY DIFFERENT t_c. E.G. IN THE GLACIER MOTION t_c CAN BE OF THE ORDER YEAR AN IN A NUCLEAR REACTOR OF THE ORDER OF MICROSECONDS.

EXAMPLE (HEAT CONDUCTION, FOR THE MODEL SEE THE NEXT SECTION):



$u \doteq u(x,t)$: TEMPERATURE AT X AND TIME

(1) $u'_t(x,t) - k\,u''_{xx}(x,t) = 0, \quad 0 \le x \le l, \quad t \ge 0$

(2) $u(x,0) \equiv 0 \quad$ for $0 < x < l$ (IC)

(3) $u(0,t) = u(l,t) = T_0 \quad$ for $t > 0$ (BC)

CHARACTERISTIC LENGTH: $l_c = l$.

CHARACTERISTIC TEMPERATURE: $u_c = T_0$

CHARACTERISTIC TIME: $t_c = l^2/k$

l HAS DIMESION L

T_0 HAS DIMENSION T

k HAS DIMENSION $\frac{L^2}{\ }$

THEREFORE

- $\bar{x} = \frac{x}{\ell}$, $\bar{t} = \frac{t}{\ell^2/k}$ AND $\bar{u} = u/T_0$.

$$u'_x = \frac{du}{dx} = \frac{du}{d\bar{x}} \cdot \frac{d\bar{x}}{dx} = T_0 \frac{d\bar{u}}{d\bar{x}} \cdot \frac{1}{\ell} = \frac{T_0}{\ell} \frac{d\bar{u}}{d\bar{x}} = \frac{T_0}{\ell} \bar{u}'_{\bar{x}}$$

$$u''_{xx} = \frac{T_0}{\ell} \cdot \frac{d^2\bar{u}}{d\bar{x}^2} \cdot \frac{d\bar{x}}{dx} = \frac{T_0}{\ell^2} \cdot \bar{u}''_{\bar{x}\bar{x}}$$

$$u'_t = \frac{du}{dt} = \frac{du}{d\bar{t}} \cdot \frac{d\bar{t}}{dt} = T_0 \frac{d\bar{u}}{d\bar{t}} \cdot \frac{d\bar{t}}{dt} = \frac{T_0 k}{\ell^2} \frac{d\bar{u}}{d\bar{t}} = \frac{T_0 k}{\ell^2} \bar{u}'_{\bar{t}}$$

REFORMULATION OF (1)-(3):

(1)' $\quad \bar{u}'_{\bar{t}}(\bar{x},\bar{t}) - \bar{u}''_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = 0, \quad \bar{t} \geq 0, \ 0 \leq \bar{x} \leq 1$

(2)' $\quad \bar{u}(\bar{x},0) = 0 \quad , \ 0 < \bar{x} < 1$

(3)' $\quad \bar{u}(0,\bar{t}) = \bar{u}(1,\bar{t}) = 1$.

THIS IS THE BVP (1)-(3) ON
<u>DIMENSIONLESS FORM.</u>

<u>REMARK:</u> IF WE CHANGE (2) TO

(2)* $\quad u(x,0) = f(x)$

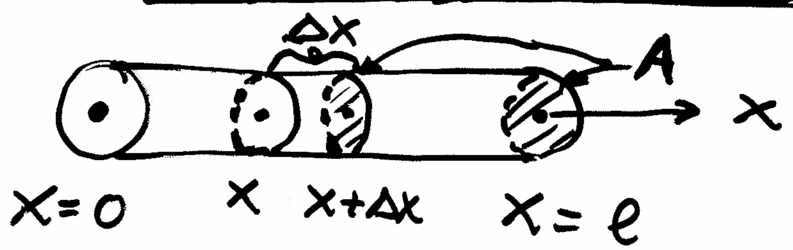
THEN ONE NATURAL CHOICE OF $\bar{u}$ COULD BE THE MEAN VALUE

$$u_c = \frac{1}{\ell}\int_0^\ell f(x)\,dx$$

BUT IN PRACTISE IT IS MORE COMMON AND EASY TO USE

$$u_c = \max_{0 \leq x \leq \ell} |f(x)| .$$

8. ON A HEAT CONDUCTION PROBLEM



C_v SPECIFIC HEAT (UNIT e.g. CAL/GRAM·DEGR

$u(x,t)$ = TEMPERATURE AT x AT TIME t.

AMOUNT OF HEAT FROM x to $x+\Delta x$:

$$C_v \tilde{\rho}\, u(\xi,t) \widetilde{A \Delta x} \qquad (x \le \xi \le x+\Delta x)$$

$\phi(x,t)$ = HEAT FLUX = THE AMOUNT OF HEAT ENERGY PER UNIT TIME FLOWING THROUGH THE FACE AT x.

a) ENERGY BALANCE EQUATION

$$\frac{\partial}{\partial t}\left(C_v \rho\, u(\xi,t) A \Delta x\right) = \phi(x,t) - \phi(x+\Delta x,t)$$

OR (BY DIVIDING BY Δx AND LETTING $\Delta x \to 0$)

$$C_v \rho A\, u'_t(x,t) = -\phi'_x(x,t).$$

b) CONSTITUTIVE RELATION:

FLUX IS PROPORTIONAL TO BOTH A AND THE TEMPERATURE GRADIENT u'_x i.e.

$$\phi(x,t) = -K A\, u'_x(x,t)$$

K = THERMAL CONDUCTIVITY $\left(\frac{CAL}{CM \cdot SEC \cdot DEGR}\right.$

a) and b) GIVES

$$C_v \rho\, u'_t(x,t) = K\, u''_{xx}(x,t)$$

$\Leftrightarrow$

$$u'_t(x,t) - k\, u''_{xx}(x,t) = 0 \qquad k = \frac{K}{c\rho} \quad \left(\frac{L^2}{}\right.$$

9 Some population models

P_0 is the population at time $t=0$.
$P=P(t)$ denotes the population at time t ($P(0)=P_0$).

Malthus model: $\frac{dP}{dt} = rP$ (*) $\boxed{P(0)=P_0}$

r = Constant growth rate

(*) has the solution

$$P(t) = P_0 e^{rt}$$

Is this possible in the long run?
(Compare with stock market and the predictions in economy etc.)

The logistics model:

$$(**) \quad \frac{dP}{dt} = rP\left(1-\frac{P}{K}\right), \quad P(0)=P_0$$

K = Carrying capacity

Scaling:

Dimensionless time: $\tau = rt$
Dimensionless population: $\mathcal{P} = \frac{P}{K}$
(or $\mathcal{P} = \frac{P}{P_0}$)

By using these dimensionless variables (**) can be written

$$(**)_s \quad \frac{K\,dP}{\frac{1}{r}\,d\tau} = rKP(1-P),\ P(0) = \frac{P_0}{K}$$

, i.e.,

$$\frac{dP}{d\tau} = P(1-P), \quad P(0) = \alpha$$

where $\alpha = P_0/K$ is dimensionless.

$(**)_s$ can be solved as usual by separating variables and we obtain that

$$P(\tau) = \frac{\alpha}{\alpha + (1-\alpha)e^{-\tau}}$$

Clearly $P(\tau) \to 1$ as $\tau \to \infty$

, i.e.,

$$P(t) \to K \quad \text{as } t \to \infty$$

$P = K$ is called an <u>attractor</u> which has the remarkable property that independent of the initial population $P(t) \to K$ as $t \to \infty$.

Problems - Lecture 1

1.* Present and discuss the Malthus and logistics models e.g. for population growths. Introduce demensionless variables and make the suitable scaling. Solve the scaled equation and decide the attractor of the original logistics equation. What is the interpretation of this?

2. A physical system is described by the law $f(E, P, A)$, where E, P and A are energy, pressure and area, respectively. Find the equivalent physical law in terms of dimensionless variables
Answer: $PA^{3/2}E = \text{const}$.

3.* A physical phenomenon is described by the quantities P, l, m, t and ρ, representing pressure, length, mass, time and density, respectively. If there is a physical law

$$f(P, l, m, t, \rho) = 0$$

relating these quantitien, then show that there is an equivalent physical law of the form

$$G(l^3\rho/m, t^6P^3/m^2\rho) = 0.$$

4. Make a suitable modelling and derive the heat (or diffusion) equation

$$\begin{cases} u'_t(x,t) - k\, u''_{xx} = 0, & 0 \le x \le \ell, \\ u(x,0) = 0, & 0 < x < \ell, \\ u(0,t) = u(\ell,T) = 100, & t > 0. \end{cases}$$

Make a suitable scaling of this equation and explain why this is so important and useful for many practical problems.

5. Let $u = u(t)$, $0 \le t \le b$, be a continuous function. If $M = \max |u(t)|$, then u can be scaled by M to obtain the dimensioless dependent variable $U = u/M$. A time scale can be taken as $t_c = M/\max|u'(t)|$, the ratio of the maximum value of the function to the maximum slope. Find M and t_c for the following functions:

a) $u(t) = A \sin \omega t$, $t > 0$,

b) $u(t) = A e^{-\lambda t}$, $t > 0$,

c) $u(t) = A t e^{-\lambda t}$, $0 < t \le 2/\lambda$.