Lecture 10

(A) Continuation of lecture 9 - p. 29-38.

(B) Dirac's deltafunction (=unit impuls)
≈ Introduction to the theory of distribution.
Green's method.

(C) Some methods and helpful tools for solving partial differential equations (PD

(D) Some words about Homogenization Theory.

(E) Advices for the future e.g. the exam!

A

10. Classification of the equilibrium points of the nonlinear system

$$(*)\qquad \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y). \end{cases}$$

Example 19 The system

$$\begin{cases} \dot{x} = 2x - 3xy \\ \dot{y} = -4y + 5xy \end{cases}$$

is nonlinear. The equilibrium points are $(0,0)$ and $\left(\frac{4}{5}, \frac{2}{3}\right)$. $\blacksquare$

Now we assume that (α_0, β_0) is an equilibrium point of $(*)$. We make a Taylor expansion around (α_0, β_0) and obtain that $(*)$ can be written

$$\begin{cases} \dot{x} = a(x-\alpha_0) + b(y-\beta_0) + f(x,y), \\ \dot{y} = c(x-\alpha_0) + d(y-\beta_0) + g(x,y). \end{cases}$$

The <u>linear approximation of $(*)$</u> in a neighbourhood of (α_0, β_0) is defined by

$$(**)\qquad \begin{cases} \dot{\xi} = a\xi + b\eta \\ \dot{\eta} = c\xi + d\eta, \end{cases} \qquad \boxed{\begin{matrix} \xi = x - \alpha_0 \\ \eta = y - \beta_0 \end{matrix}}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} P'_x(\alpha_0,\beta_0) & P'_y(\alpha_0,\beta_0) \\ Q'_x(\alpha_0,\beta_0) & Q'_y(\alpha_0,\beta_0) \end{pmatrix}.$$

* We note that also (**) has the equilibrium point $(x,y)=(\alpha_0,\beta_0)$.

Example 13: The linear approximation of the system
$$\begin{cases}\dot{x} = 2x - 3xy,\\ \dot{y} = -4y + 5xy,\end{cases}$$
in a neighbourhood of $(0,0)$ is
$$\begin{cases}\dot{x} = 2x,\\ \dot{y} = -4y.\end{cases}$$
$$\left(A = \begin{pmatrix} P'_x(0,0) & P'_y(0,0) \\ Q'_x(0,0) & Q'_y(0,0)\end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -4\end{pmatrix}\right)$$

In order to avoid technical difficulties we assume that

(a) $\det A = \begin{vmatrix} a & b \\ c & d\end{vmatrix} \neq 0.$

(This condition guarantees that (α_0,β_0) is an isolated equilibrium point of (**)

(b) $f(x,y) = o\left(\sqrt{(x-\alpha_0)^2+(y-\beta_0)^2}\right)$, as $(x,y)\to(\alpha_0,\beta_0)$,

$g(x,y) = o\left(\sqrt{(x-\alpha_0)^2+(y-\beta_0)^2}\right)$, as $(x,y)\to(\alpha_0,\beta_0)$

(These conditions guarantee that (α_0,β_0) is an isolated equilibrium point of (*

The following useful Theorem holds:

Theorem 1: Let (α_0, β_0) be an equilibrium point of $(*)$ and $(**)$ subject to the assumptions (a) and (b).

(α_0, β_0) is an equilibrium point for $(*)$ of the same type as for $(**)$ in the following cases:

(i) The eigenvalues of A are real distinct, and have the same sign. (node)

(ii) The eigenvalues of A are real, distinct, and have opposite signs (saddle)

(iii) The eigenvalues of A are complex but not purely imaginary. (spiral)

Example 14: The system
$$\begin{cases} \dot{x} = 2x - 3xy \\ \dot{y} = -4y + 5xy \end{cases}$$
has a saddle at $(0,0)$.

Proof: $(0,0)$ is an equilibrium point and the matrix A of the linear approximation of this system is
$$A = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$
with the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -4$. Thus the statement follows from Theorem 1(ii

Example 15: The system
$$\begin{cases} \dot{x} = -2x + 3y + xy \\ \dot{y} = -x + y - 2xy^3 \end{cases}$$
has a stable spiral at $(0,0)$.

Proof: $(0,0)$ is an equilibrium point and the linear approximation of this system has the matrix
$$A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$$
with the eigenvalue $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$. Therefore the statement follows from Theorem 1 (iii).

* We remark that Theorem 1 fails in all cases where (i)-(iii) do not hold. E.g. for the purely imaginary case we have the following counterexample:

Consider (1) $\begin{cases} \dot{x} = -y - x^3, \\ \dot{y} = x. \end{cases}$

$(0,0)$ is a spiral of (1) (Prove that)

The corresponding linear approximation is
$$(2) \begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases}$$
has the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with the eigenvalues $\pm i$ so that $(0,0)$ is a center of (2).

11 Construction of a general phase diagram of the non-linear system

$$(*)\quad \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y). \end{cases}$$

(a) Find the equilibrium points by solving the system

$$\begin{cases} P(x,y) = 0, \\ Q(x,y) = 0. \end{cases}$$

(b) For each equilibrium point (α_0, β_0) we consider the linear approximation of $(*)$:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \boxed{\begin{array}{l} \xi = x - \alpha_0 \\ \eta = y - \beta_0 \end{array}}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} P'_x(\alpha_0,\beta_0) & P'_y(\alpha_0,\beta_0) \\ Q'_x(\alpha_0,\beta_0) & Q'_y(\alpha_0,\beta_0) \end{pmatrix}$$

Use Theorem 1 for the cases it can be applied!

(c) Sketch the phase diagram close to the equilibrium points by using the relation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

and $(*)$ to obtain the direction.

(d) Sketch some paths "far" from the equilibrium points

Remark: In (c) and (d) we can use e.g. REGSIM.

Example 16: Consider the system

$$\begin{cases} \dot{x} = x - xy, \\ \dot{y} = x^2y^2 - 4y. \end{cases}$$

a) Calculate all equilibrium points.

b) Classify these equilibrium points.

Solution (a): We have to solve the system

$$\begin{cases} (1) \quad P(x,y) = x - xy = 0 \\ (2) \quad Q(x,y) = x^2y^2 - 4y = 0 \end{cases}$$

(1) $\Leftrightarrow$ $x(1-y) = 0 \Leftrightarrow x = 0$ or $y = 1$

$x = 0$ in (2) gives that $y = 0$

$y = 1$ in (2) gives $x = 2$ or $x = -2$

Thus the equilibrium points are

$(0,0)$, $(2,1)$ and $(-2,1)$

(b) Consider

$$A = \begin{pmatrix} P'_x & P'_y \\ Q'_x & Q'_y \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ 2xy^2 & 2x^2y-4 \end{pmatrix}$$

$(0,0)$ $A = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -4-\lambda \end{vmatrix} = -(1-\lambda)(4+\lambda) = 0$$

with the solutions $\lambda_1 = 1$ and $\lambda_2 = -4$.

Thus, by Theorem 1, we conclude that

* $(0,0)$ is a saddle.

$(2,1)$ $A = \begin{pmatrix} 0 & -2 \\ 4 & 4 \end{pmatrix}$. The characteristic equation is

$$\begin{vmatrix} 0-\lambda & -2 \\ 4 & 4-\lambda \end{vmatrix} = (0-\lambda)(4-\lambda)+8=0 \text{ i.e.}$$

$\lambda^2 - 4\lambda + 8 = 0$ with the solutions $\lambda_1 = +2 + 2i$ and $\lambda_2 = 2 - 2i$.

Hence, by Theorem 1, we conclude that

* $(2,1)$ is an unstable spiral.

$(-2,1)$ In a similar way we find that ..

* $(-2,1)$ is ?

Homework: Control the details!

12. On the possibility to have closed paths of a system

$$(*)\quad \begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$$

A solution $x = x(t)$, $y = y(t)$ of $(*)$ is periodic if neither $x(t)$ nor $y(t)$ is constant and there exists a positive number T such that

$$x(t+T) = x(t) \text{ and } y(t+T) = y(t) \ \forall t.$$

The smallest such T is called the period.

Example 17: (C.f. Example ?). The system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

has 2π periodic solutions corresponding to the closed paths (circles)

$$x^2 + y^2 = c^2$$

in the phase plane.

For general nonlinear systems the existence of closed paths may be difficult to decide

Theorem A (Benedixon) If $P'_x + Q'_y$ has the same sign in a region of the phase plane, then the system $(*)$ <u>cannot</u> have a closed path in that region.

Example 18: The system
$$\begin{cases} \dot{x} = y^2x - x, \\ \dot{y} = x^2y + 2y, \end{cases}$$
can not have a closed path in $\mathbb{R}^2$.

Proof: $P'_x + Q'_y = y^2 - 1 + x^2 + 2 > 0 \quad \forall (x,y) \in \mathbb{R}^2$.
The statement follows by using Benedixon's negative criterion (Theorem A)

Theorem B (Poincaré) A closed path of the system (*) surrounds at least one critical point of (*).

Theorem C (Poincaré–Benedixon): Let R_0 be a closed bounded region in the plane containing no critical points of (*). If C is a path of (*) that lies in R_0 for some t_0 and remains in R_0 for all $t > t_0$, then C is either a closed path or it spirals towards a closed path as $t \to \infty$.

* Theorem C implies that in the plane the only attractors are closed curves or critical points.

* This is NOT true in three dimensions. Here we have even <u>strange attractors</u> etc....

13. On bifurcations of the parameter-dependent system

$$\begin{cases} \dot{x} = P(x, y, \mu), \\ \dot{y} = Q(x, y, \mu). \end{cases}$$

Example 19: Consider the system

$$\begin{cases} \dot{x} = x + \mu y, \\ \dot{y} = x - y, \end{cases}$$

where μ is a real number. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & \mu \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - (1+\mu) = 0$$

Three cases

1° $\mu > -1$ $\quad \lambda_1 = \sqrt{\mu+1}, \lambda_2 = -\sqrt{\mu+1}$.

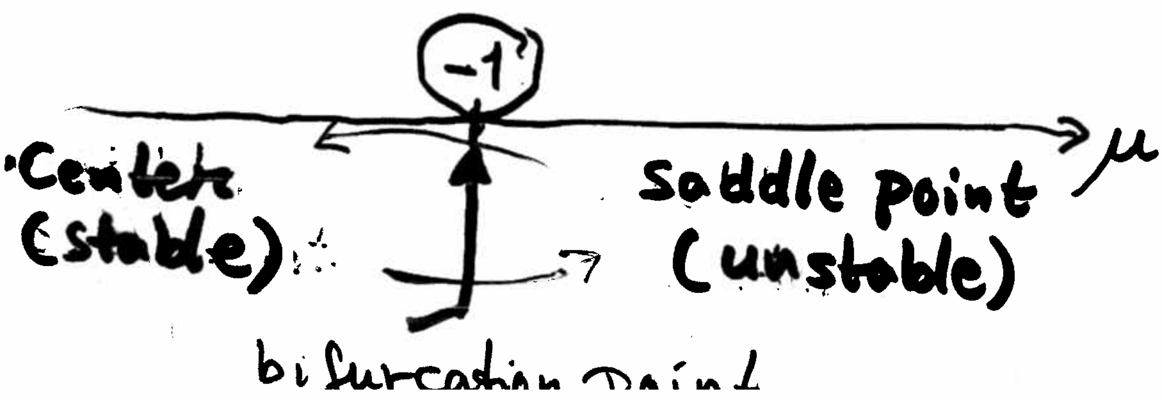
$\therefore$ $(0,0)$ is a saddle point

2° $\mu = -1$. Here the system is $\begin{cases} \dot{x} = x - y \\ \dot{y} = x - y \end{cases}$

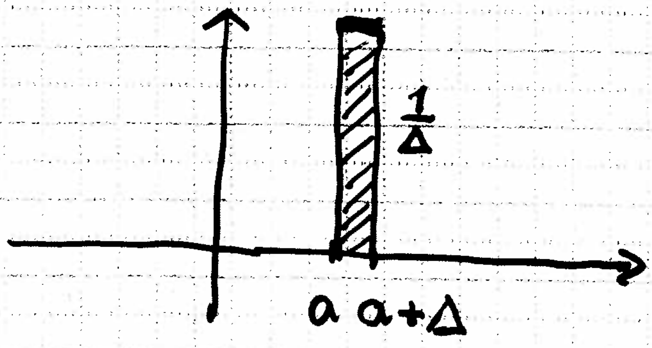
All points (a,a) are equilibrium points.

3° $\mu < -1$ $\quad \lambda_1 = i\sqrt{-1-\mu}, \lambda_2 = -i\sqrt{-1-\mu}$

$\therefore$ $(0,0)$ is a center



(5) Dirac's deltafunction (unit impuls)



Consider

$$S_a(x) = \begin{cases} \frac{1}{\Delta} & a \le x < a+\Delta \\ 0 & \text{elsewhere} \end{cases}$$

We note that $\int_{-\infty}^{\infty} S_a(x)\,dx = 1$.

Let $\Delta \to 0$ and $S_a(x) \to \delta_a(x)$.

$\delta_a(x)$ is the simplest case of generalized function (distribution) and it is called
Dirac's delta function (By Mathematicians)
unit impuls (By Engineers)

The following holds:

* $\delta_a(x) = 0 \quad x \neq a,$
* $\phi(x)\delta_a(x) = \phi(a)\delta_a(x)$ all continuous $\phi(x)$
* $\int_{-\infty}^{\infty} \delta_a(x)\,dx = 1$
* $\int_{-\infty}^{\infty} \delta_a(x)\phi(x)\,dx = \phi(a)$
* $\frac{d}{dx}\Theta_a(x) = \delta_a(x)$ and $\Theta_a(x) = \int_{-\infty}^{x} \delta_a(t)\,dt$

where $\Theta_a(x)$ is the unit-step function

$$\Theta_a(x) = \begin{cases} 0, & x < a \\ 1, & x \ge a \end{cases}$$

Green's function

Mathematically: It is the Kernel of the integral operator corresponding to the differential operator

Physically: It is the response of a system when a unit point source is applied to the system ("the unit impulse answer")

Greens method:

"Mathematically"

1): solve diff. ekv.

(*) $Du = f$

$u(0)=0, \quad u(1) = 1.$

2) Solve

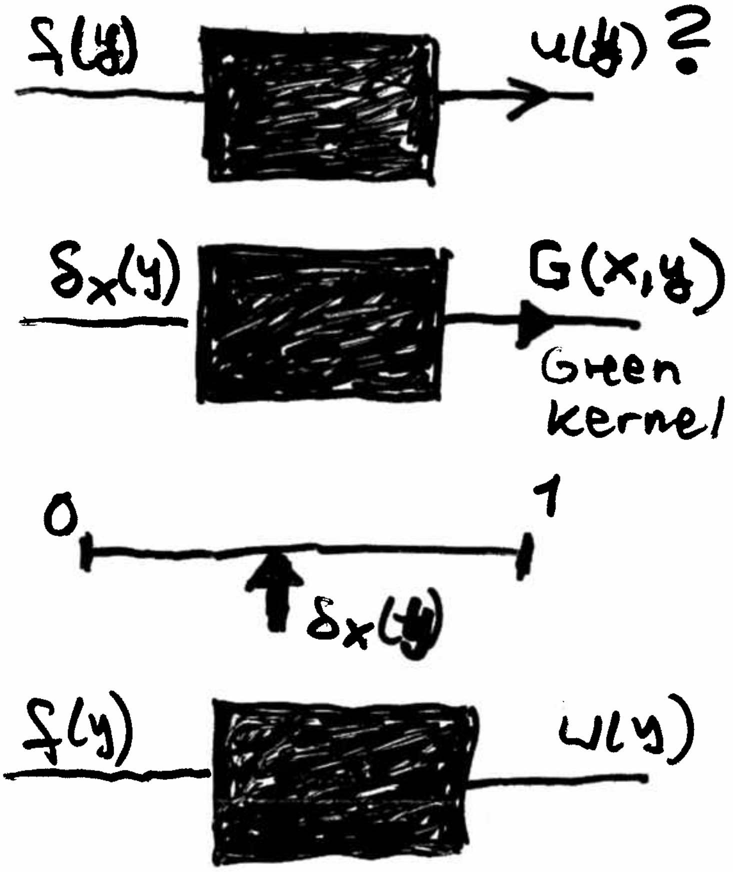
$Du = \delta_x$

sol: $G(x,y)$

3) Sol of (*)

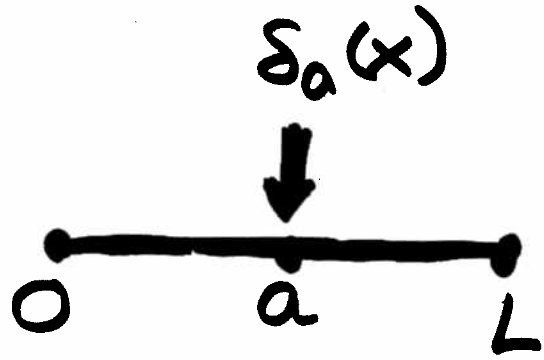
$$u(x) = \int_0^1 G(x,y) f(y)\, dy$$

"Techniquelly"

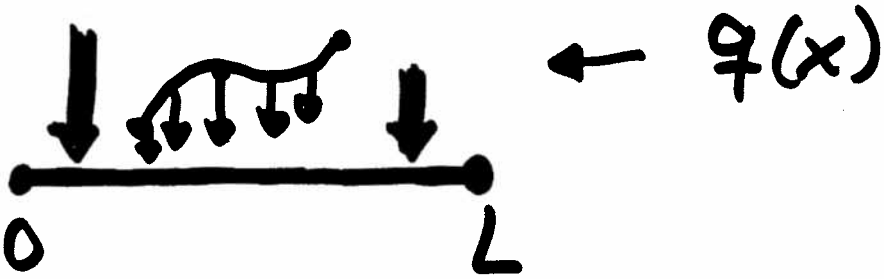


$$u(x) = \int^1 G(x,y) f(y)\, dy$$

Example: (Beam Analysis)



$$W(x) = K(x,a)\,, \quad 0 \le a \le L$$



$$W(x) = \int_0^L K(x,a)\, q(a)\, da$$

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right) = \delta_a(x)$$

$$\frac{d}{dx}\left(EI(x)\frac{d^2w(x)}{dx^2}\right) = \Theta_a(x) + A$$

$$EI(x)\frac{d^2w(x)}{dx^2} = (x-a)\Theta_a(x) + Ax + B$$

Assume, for simplicity, that $I(x)$ is constant $= I$. Then

$$EI\frac{dw(x)}{dx} = \frac{(x-a)^2}{2}\Theta_a(x) + A\frac{x^2}{2} + Bx + C$$

$$EIw(x) = \frac{(x-a)^3}{6}\Theta_a(x) + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D$$

Thus

$$K(x,a) = \frac{1}{EI}\frac{(x-a)^3}{6}\Theta_a(x) + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D$$

where the constants A, B, C, D can be calculated by using the boundary conditions ("balkens inspänningsvillkor).

The solution for any $q(x)$ is

$$w(x) = \int_0^L K(x,a)\,q(x)\,dx.$$

A lot of details, examples and a computer program can be found in a Master Thesis of Thomas Strömberg and Peter Wall.

© Some methods and helpful tools for solving partial differential equations

A Numerical methods

1°. Finite Element Method (FEM)

2°. Boundary Element Method (BEM)

3°. Finite difference Methods

4. Wavelets programs

B. Analytical methods

1°. Transformation methods. (Fourier-transform, Laplace transform, fast Fourier transform, Hankel transform, Z-transform, Wavelet transform, etc.)

2°. Separation of variables.

* In simple cases we only need the usual theory for Fourier series

* In other cases we use the theory of generalized Fourier series (e.g. for Bessel equations etc.)

3°. Green's method (≈ the method to "test" the system (equation) with unit impulses)

4° The method of characteristics for first order equations (Chapter 6)

5° Suitable substitutions (by e.g. knowing additional Physical facts) (e.g. $u = f(x-ct)$, $u = A\, e^{i(kx-\omega t)}$, etc.)

6° Variational method's

via Euler's equations

7° Perturbation Methods

regular perturbation, singular perturbation, matching, etc., ...

8° Functional Analytic Methods

9° Interpolation Methods

In many situations it is very powerfu to use suitable combinations of the methods above. For example for PDE with extremely varying coefficients we have a method

The Homogenization method

to solve such problems.

(*) L-E Persson, L Persson, N. Svanstedt and J. Wylle
The Homogenization Method - An Introduction
Studentlitteratur Publ. 1993.

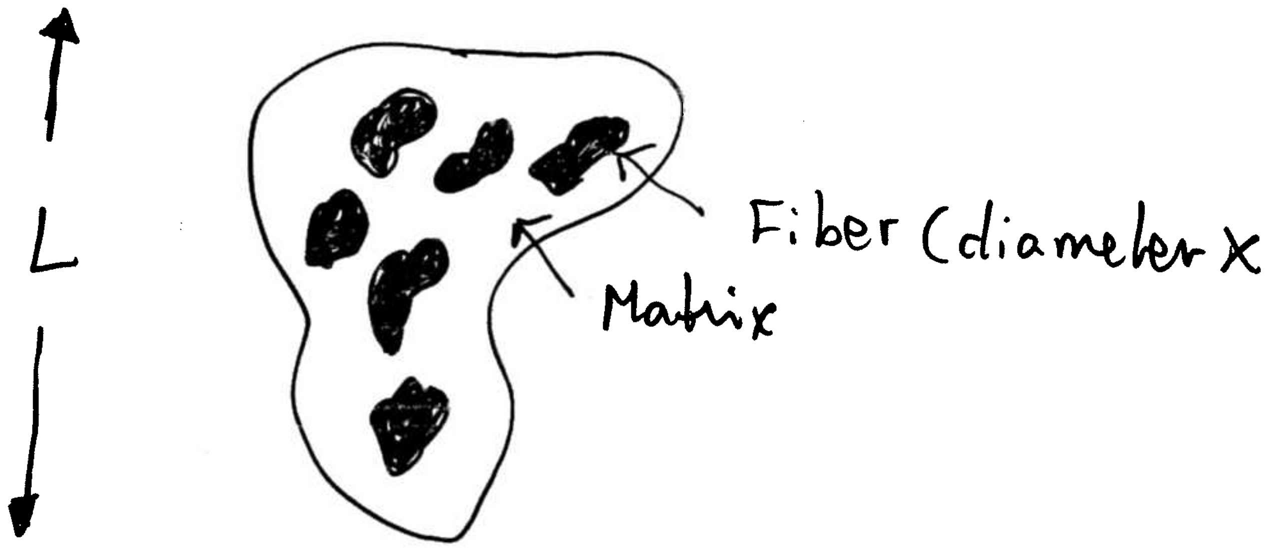
(*) Ph.D. Theses of {Nils Svanstedt, Peter Wal
[redacted] Anders Holmbom

(Useful tools: $A_1, B_7, B_8, B_9, \ldots$) Dag Lukkassen

(D) VII The Homogenization method

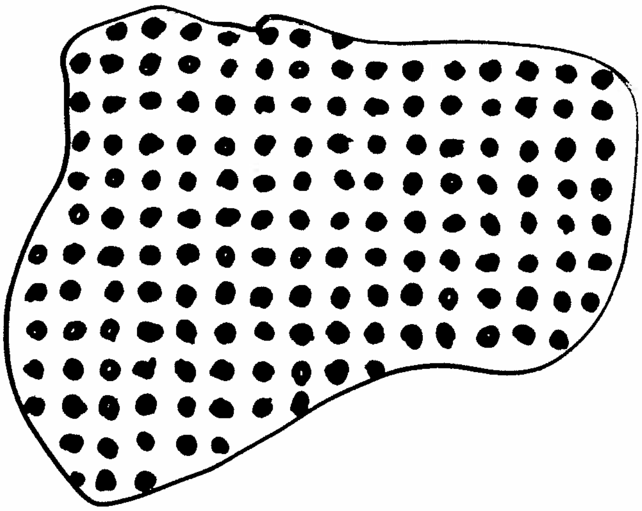
A COMPOSITE WITH 2 MATERIALS

1.

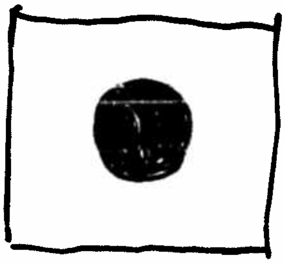


Use e.g. some numerical (FEM) or analytical method directly

2



$L >> x$. Some homogenization method ought to be used.



UNIT CELL

Local scale:

ε l

Ex 1: THE AGH inequality - elementary form (1)
(= Introduction to homogenization theory).

Heat conduction

a)

Heat conduction number λ_1
Heat conduction number λ_2

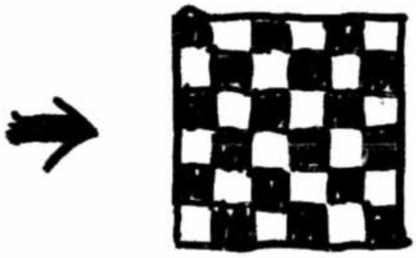
Effective heat conduction number

$$H = \frac{2}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}$$ (Harmonic mean)

b)

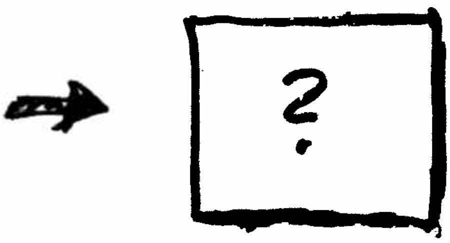
$$A = \frac{\lambda_1+\lambda_2}{2}$$
(Arithmetic mean)

c)



$$G = \sqrt{\lambda_1\lambda_2}$$
(Geometric mean)

$$P_\alpha = \left(\frac{1}{2}\sum_{i=1}^{2}\lambda_i^\alpha\right)^{1/\alpha}, \alpha \neq 0$$



$P_\alpha \uparrow$ (Power means)

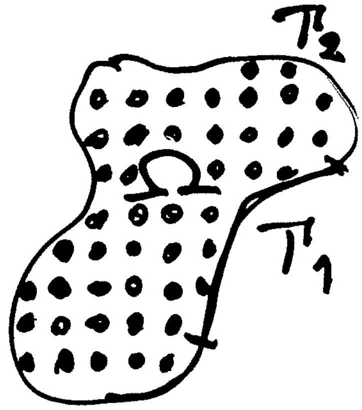
Remark: Note that the above clearly illustrates that $H \leq G \leq A$

More generally: P_α is increasing in α

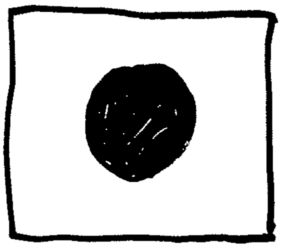
(This we will prove later on)

A model example

$$(1)\quad \begin{cases} -\dfrac{\partial}{\partial x_i}\left(a_{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u^{\varepsilon}}{\partial x_j}\right) = f\left(\dfrac{x}{\varepsilon}\right) \text{ in } \Omega \\ u^{\varepsilon}=0 \text{ on } \Gamma_1 \text{ and } a_{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u^{\varepsilon}}{\partial x_j}n_i = g \text{ on } \Gamma_2 \end{cases}$$



Global variable x
local variable $y = \frac{x}{\varepsilon}$



Y-cell

The cell-problem:

$$(2)\quad +\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_i}\right)\chi^j = \frac{\partial}{\partial y_i}\left(a_{ij}(y)\right)$$

χ^j is Y-periodic

The homogenized equation ($\varepsilon \to 0$)

$$(3)\quad -q_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} = \langle f_0 \rangle$$

where the homogenized coefficients are

$$(4)\quad q_{ik} = \left\langle a_{ik} - a_{ij}\frac{\partial \chi^k}{\partial y_j}\right\rangle$$

$\langle f \rangle = 1 \int f(y)dy$

A homogenization procedure (for solving (1):

1. Solve the cell problem (2)
2. Insert this solution into (4) and solve (3) (e.g. numerically)
3. The local variations are obtained by using (2) once more.

Remark: An elementary introduction to homogenization theory and its applications can be found in

[*] The Homogenization Method. An Introduction, Studentlitteratur 1993.
Authors: L.E. Persson, L. Persson, N. Svanstedt and J. Wyller

For first chapter see Appendix!

Problems - Lecture 10

1.* (a) Define Dirac's delta function ("the unit impulse"), Heavisidefunctione ("the unit step function") and the Green kernel ("the unit impulse answer")

(b) Describe shortly Green's method for solving differential equations ("in-out signal relations").
What is the crucial assumption so that this method works?

(c) Give an example of a ~~technical~~ or natural scien problem which can be solved in this way.

2.* (a) Determine the nature and stability properties of the critical points of the following dynamical system

$$\begin{cases} \dot{x} = x + y - 2x^2 \\ \dot{y} = -2x + y + 3y^2. \end{cases}$$

(b) Consider the dynamical system:

$$\begin{cases} \dot{x} = 4x + 4y - x\,(x^2 + y^2) \\ \dot{y} = -4x + 4y - y\,(x^2 + y^2). \end{cases}$$

Show that there is a closed path in the region $1 \leq x^2 + y^2 \leq 9$.

3. Determine the nature and stability properties of the critical points of the following dynamical system:

$$\begin{cases} \dot{x} = -x - y - 3x^2 y \\ \dot{y} = -2x - 4y + y\sin x. \end{cases}$$

4.* Consider the dynamical system:

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = x^2 y^2 - 4y. \end{cases}$$

(a) Calculate all equilibrium points.

(b) Classify each of these equilibrium points.

5. = Problem 4 from Lecture 9.

6. = Probem 6 from Lecture 9.

7. a) Describe briefly some problem which can be handled by the homogenization method (but which is difficult by other methods).

b) Describe briefly the ideas behind the homogenization method and some concrete homogenizatio algorithm.

(This problem can also be used as *) - problem)

A1

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The Homogenization Method

An Introduction

Studentlitteratur