2 Perturbation methods

1. Example:

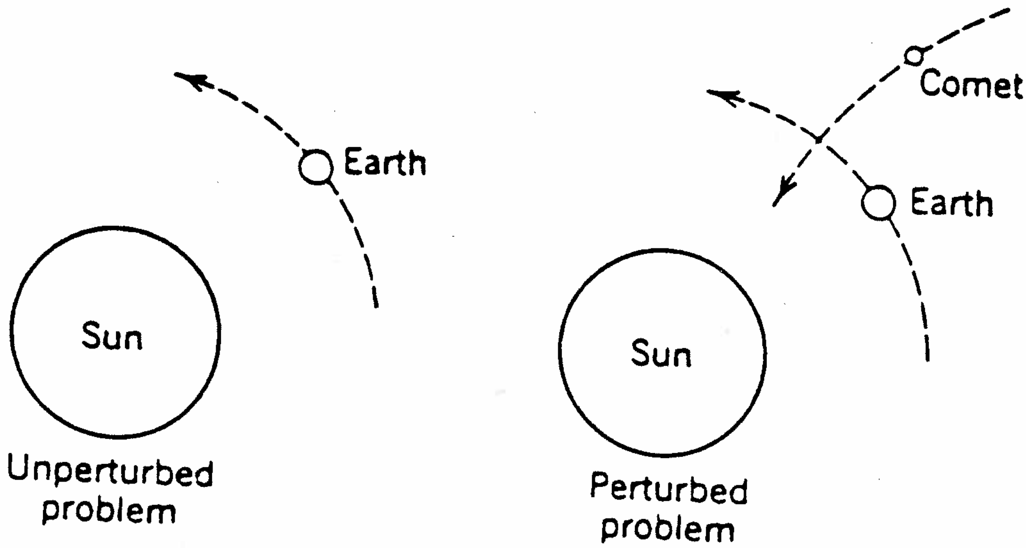


Figure 2.1. Unperturbed and perturbed system.

* The unperturbed problem is governed by

$$(1) \qquad m\ddot{y} = F_{sun} \quad \text{(Newton's law)}$$

* The perturbed problem is governed by

$$(2) \qquad m\ddot{y} = F_{sun} + \varepsilon F_{comet}$$

It seems reasonable that the solution of (2) can be of the form

$$y(t) = \underbrace{y_0(t)}_{\text{Solution of (1)}} + \underbrace{\varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots}_{\text{Correction terms}}$$

2. The main idea.

Consider the second order differentia[l] equation

$$(*) \qquad F(t, y, \dot{y}, \ddot{y}; \varepsilon) = 0 , \quad 0 \le t \le t_0,$$

where

$$\varepsilon << 1 .$$

Try to solve (*) by inserting the <u>perturbation series</u>

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

and find $y_0(t)$, $y_1(t)$, $y_2(t)$; say, and use e.g.

$$y_{appr.} = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$$

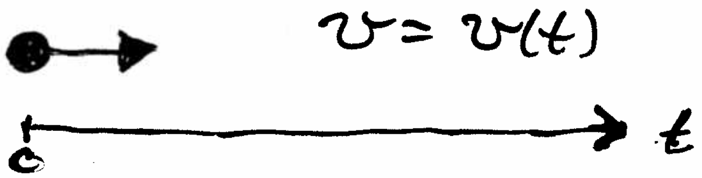
as a good approximation of the solution of (*). Here $y_0(t)$ is the <u>leading order term</u> which is the solution of the <u>unperturbed problem</u>

$$(**) \qquad F(t, y, \dot{y}, \ddot{y}; 0) = 0 , \quad 0 \le t \le t_0,$$

Here $\varepsilon y_1(t), \varepsilon^2 y_2(t), \dots$ are higher order corrections which usually are "small".

<u>Remark</u>: In many cases (**) is easy to solve exactly (e.g. it is linear with constant coefficients)

3. Motion in a nonlinear resistive medium



$$(*) \qquad m\frac{dv}{dt} = -av + bv^2 \quad , v(0) = V_0 \, , b << a$$

Remark: If $b = 0$, then $v(t) = V_0 e^{-\frac{a}{m}t}$.

SCALING: Introduce dimensionless variables

$$y = \frac{v}{V_0} \, , \quad x = \frac{t}{m/a} \, ,$$

$$\boxed{\begin{aligned} dv &= V_0 \, dy \\ dt &= \frac{m}{a} dx \end{aligned}}$$

and (*) becomes

$$(**) \quad \begin{cases} \dfrac{dy}{dx} = -y + \varepsilon y^2 \, , \quad x > 0, \\ y(0) = 1, \end{cases}$$

where $\varepsilon = \dfrac{bV_0}{a} << 1$.

Remark 1: The unperturbated problem

$$\frac{dy}{dx} = -y \quad , \, y(0) = 1$$

has the solution $y(t) = e^{-t}$.

Remark 2: (**) has, in fact, the exact solution

$$y_{exact} = \frac{e^{-x}}{1 + \varepsilon(e^{-x} - 1)}$$

Hint: make the substitution $z = 1/y$!

4. Solution of (**) by the perturbation method

$$(**) \qquad \frac{dy}{dx} = -y + \varepsilon y^2, \quad y(0) = 1.$$

Inserting

$$y = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

into (**) gives

- $y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \dots = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) + \varepsilon (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)^2 =$

$$= -y_0 + \varepsilon(-y_1 + y_0^2) + \varepsilon^2(-y_2 + 2 y_0 y_1) + \dots$$

and

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \dots = 1, \text{ i.e.,}$$

- $y_0(0) = 1, \; y_1(0) = 0, \; y_2(0) = 0, \dots$.

Thus we can determine $y_0, y_1, y_2, \dots$ consequtively as follows:

ε^0: $y_0' = -y_0, \; y_0(0) = 1 \quad \therefore \boxed{y_0 = e^{-x}}$

ε^1: $y_1' = -y_1 + y_0^2, \; y_1(0) = 0$, i.e.,

$y_1' + y_1 = e^{-2x}, \; y_1(0) = 0. \quad \therefore \boxed{y_1 = e^{-x} - e^{-2x}}$

ε^2: $y_2' = -y_2 + 2 y_0 y_1, \; y_2(0) = 0$, i.e.,

$y_2' + y_2 = 2(e^{-2x} - e^{-3x}). \quad \therefore \boxed{y_2 = e^{-x} - 2e^{-2x} + e^{-3x}}$

$$y_{appr} = e^{-x} + \varepsilon(e^{-x} - e^{-2x}) + \varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x})$$

5. A comparison with the exact solution.

$$y_{exact} = \frac{e^{-x}}{1+\varepsilon(e^{-x}-1)}$$

$\boxed{\text{Geometric series} \quad 1-k+k^2-k^3+\dots = \frac{1}{1+k}}$

$\because\ y_{exact} = e^{-x}\left(1-\varepsilon(e^{-x}-1)+\varepsilon^2(e^{-x}-1)^2+\dots\right)$ i.e.

$$y_{exact} = e^{-x}+\varepsilon(e^{-x}-e^{-2x})+\varepsilon^2(e^{-x}-2e^{-2x}+e^{-3x})+\dots$$

We remember that

$$y_{appr.} = e^{-x}+\varepsilon(e^{-x}-e^{-2x})+\varepsilon^2(e^{-x}-2e^{-2x}+e^{-3x})$$

Therefore the error.

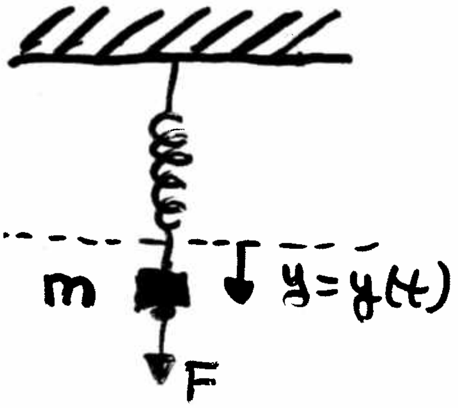
$$y_{exact}-y_{appr.} = m_1(x)\,\varepsilon^3+m_2(x)\,\varepsilon^4+\dots$$

for some $m_1(x), m_2(x)$ (which we can compute!)

We say that the error (E) is of the order ε^3 and write $E = O(\varepsilon^3)$.

$\uparrow$ ordo

6. A Nonlinear Oscillator



Assume that $F = -ky - ay^3$

$a \ll k$

Newton's second law gives

$$m\frac{d^2y}{dt^2} = -ky - ay^3$$

Moreover, $y(0) = A$, $\frac{dy}{dt}(0) = 0$.

By "scaling" this problem as before we have to solve the equation

$$(*) \quad \begin{cases} \ddot{u} + u + \varepsilon u^3 = 0, & t > 0, \\ u(0) = 1, \ \dot{u}(0) = 0. \end{cases}$$

($\varepsilon = \frac{aA^2}{k} \ll 1$ is a dimensionless parameter)

(*) is the well-known <u>Duffing's equations</u>

Our next step is to try to solve (*) by using our introduced perturbation method.

Put

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots\cdots .$$

Inserting $u(t)$ into (*) gives

$$\begin{cases} (\ddot{u}_0 + \varepsilon \ddot{u}_1 + \cdots) + (u_0 + \varepsilon u_1 + \cdots) + \varepsilon(u_0 + \varepsilon u_1 + \cdots)^3 = 0, \\ u_0(0) + \varepsilon u_1(0) + \cdots = 1, \ \dot{u}_0(0) + \varepsilon \dot{u}_1(0) + \cdots = 0 \end{cases}$$

Equating in powers of ε gives

ε^0: $\ddot{u}_0 + u_0 = 0$, $u_0(0) = 1$, $\dot{u}_0(0) = 0$

$\therefore$ $u_0(t) = \cos t$

ε^1: $\ddot{u}_1 + u_1 + u_0^3 = 0$, $u_1(0) = 0$, $\dot{u}_1(0) = 0$, i.e.,

$\ddot{u}_1 + u_1 = -\cos^3 t$, $u_1(0) = 0$, $\dot{u}_1(0) = 0$.

$\ddot{u}_1 + u_1 = -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t$, $u_1(0) = \dot{u}_1(0) = 0$

The solution is

$$u_1 = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8} t \sin t$$

$$\therefore \quad u_{appr.} = \cos t + \varepsilon\left\{\frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8} t \sin t\right\}.$$

We note that

i) the leading term "$\cos t$" seems OK.

) If $t \leq T_0$ and ε is "small" then the correction term is "small".

) If we permit t to be large ($t \to \infty$) the correction term can be large even if ε is "small".

mark. The trouble in (3) depends on the <u>Secular term</u> $-\frac{3}{8} t \sin t$.

Our next step is to introduce a method to avoid this trouble

7. The Poincaré-Lindstedt method

(= A method to avoid secular terms).

Introduce a distorted time τ
$$\tau = \omega t,$$
where

(1) $$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$$

and put

(2) $$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \dots$$

Example: We recall Duffing's equation

(*) $$\begin{cases} \ddot{u} + u + \varepsilon u^3 = 0, & t > 0 \\ u(0) = 1, & \dot{u}(0) = 0 \end{cases}$$

By performing the scale transformation $\tau = \omega t$ (*) becomes

(**) $$\begin{cases} \omega^2 u'' + u + \varepsilon u^3 = 0, & \tau > 0, \\ u(0) = 1, & \dot{u}(0) = 0. \end{cases}$$

$(u = u(\tau))$)

$$\boxed{\begin{aligned} \dot{u} &= \frac{du}{dt} = \frac{du}{d\tau}\underbrace{\frac{d\tau}{dt}}_{\omega} = \omega u' \\ \ddot{u} &= \frac{d^2u}{dt^2} = \omega\frac{d^2u}{d\tau^2}\frac{d\tau}{dt} = \omega^2 u'' \end{aligned}}$$

Substituting (1) and (2) into (**) gives

$$(1+2\omega_1\varepsilon+\cdots)(u_0''+\varepsilon u_1''+\cdots)+(u_0+\varepsilon u_1+\cdots)$$
$$+\varepsilon(u_0^3+3\varepsilon u_0^2 u_1+\cdots)=0$$

and

$$u_0(0)+\varepsilon u_1(0)+\cdots=1,\quad u_0'(0)+\varepsilon u_1'(0)+\cdots=0.$$

Equating in powers of ε gives

ε^0 $\quad u_0''+u_0=0,\ u_0(0)=1,\ u_0'(0)=0$

$$u_0(\tau)=\cos\tau$$

ε^1 $\quad u_1''+u_1=-2\omega_1 u_0''-u_0^3,\ u_1(0)=u_1'(0)=0,\ldots$

$$u_1''+u_1=2\omega_1\cos\tau-\cos^3\tau,\ u_1(0)=u_1'(0)=0$$

$$u_1''+u_1=\left(2\omega_1-\frac{3}{4}\right)\cos\tau-\frac{1}{4}\cos 3\tau$$

! By choosing $\omega_1=\frac{3}{8}$ we can avoid the secular term! Then

$$u_1''+u_1=-\frac{1}{4}\cos 3\tau$$

$$u_1(\tau)=\frac{1}{32}(\cos 3\tau-\cos\tau)$$

Summing up we find that a first order perturbation solution of Duffing's equation (*) is

$$u(\tau)=\cos\tau+\frac{1}{32}\varepsilon(\cos 3\tau-\cos\tau)+\cdots$$

where

$$\tau=t+\frac{3}{8}\varepsilon t+\cdots$$

Remark Poincaré - Lindstedt's method works for some (not all) equations of the form

$$y'' + \omega_0^2 y = \varepsilon F(t, y, y') \quad \varepsilon \ll 1$$

The problem is when also the right hand has the frequency ω_0 in some step. We will then not succeed with the particular solution

$$y_p = A \cos \omega_0 t + B \sin \omega_0 t \; \ldots\ldots$$

but must try with

$$y_p = A t \cos \omega_0 t + B t \sin \omega_0 t$$

which leads to secular terms.
The problem can be avoided (sometimes) with P-C-L method by putting

$$\tau = (\omega_0 + \omega_1 \varepsilon + \cdots) t$$

and solve the corresponding equation as usual (see examples above).

8 The Ordo notations

We write
$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as } \varepsilon \to 0$$
if
$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0 .$$
We write
$$f(\varepsilon) = O(g(\varepsilon)) \quad \text{as } \varepsilon \to 0$$
if there exists a positive constant M such that
$$|f(\varepsilon)| \le M |g(\varepsilon)| .$$
in some neighbourhood of zero.

Example: $\varepsilon = o(\varepsilon^{1/3})$

$$\sin \varepsilon = O(\varepsilon) , \quad \sin \varepsilon \neq o(\varepsilon)$$

$$\left(\begin{array}{c} \sin \varepsilon = \varepsilon - \frac{\varepsilon^3}{3} + \dots \\ \left| \frac{\sin \varepsilon}{\varepsilon} \right| = \left| 1 - \frac{\varepsilon^2}{3} + \dots \right| \le 1 , \\ \text{for sufficiently small } \varepsilon > 0. \end{array} \right)$$

9. Failure of Regular perturbation.

Example: Consider

$$(*) \quad \begin{cases} \varepsilon y'' + (1+\varepsilon) y' + y = 0 \quad , 0<t<1, \ 0<\varepsilon<<1 \\ y(0)=0 \ , \ y(1)=1. \end{cases}$$

Substitution of

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

into (*) gives

$$\varepsilon (y_0'' + \varepsilon y_1'' + \cdots) + (y_0' + \varepsilon y_1' + \cdots) +$$

$$+ (\varepsilon y_0' + \varepsilon^2 y_1' + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = 0$$

and

$$y_0(0) + \varepsilon y_1(0) + \cdots = 0 \ , \ y_0(1) + \varepsilon y_1(1) + \cdots = 1$$

As before, by equating into powers of ε, we obtain that

ε^0: $\quad y_0' + y_0 = 0 \quad , \ y_0(0) = 0, \ y_0(1) = 1$

we note that

(1) the general solotion $y_0 = c e^{-t}$

(2) $y_0(0) = 0 \Rightarrow y_0 \equiv 0$ but then the condition $y_0(1) = 1$ is <u>NOT</u> satisfied

(3) $y_0(1) = 1 \Rightarrow y_0 = e^{1-t}$ but then the condition $y_0(0) = 0$ is <u>NOT</u> satisfied.

$\therefore$ Regular perturbation fails !!

10. Inner and outer approximations

$$(*)\ \begin{cases} \varepsilon y'' + (1+\varepsilon) y' + y = 0, & 0 < t < 1, \quad 0 < \varepsilon << 1, \\ y(0) = 0, \quad y(1) = 1. \end{cases}$$

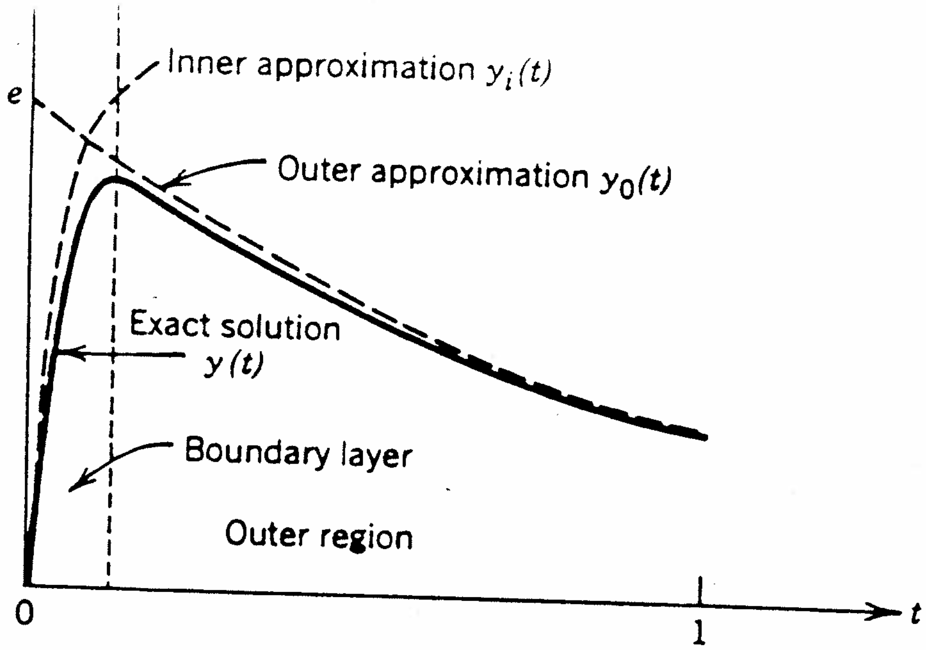


Figure 2.4. Schematic of exact solution compared to inner and outer approximation for a fixed value of ε.

Exact solution of $(*)$:

$$y(t) = c\left(e^{-t} - e^{-t/\varepsilon}\right), \quad c = 1/\left(e^{-1} - e^{-1/\varepsilon}\right).$$

The unperturbed probem of $(*)$

$$y' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

- If we only require that $y(1) = 1$ we obtain the <u>outer approximation</u>

$$y_0(t) = e^{1-t}.$$

- If t is "small" (t is in the <u>boundary layer</u>), then

$$y(t) \approx e - e^{1-t/\varepsilon} = y_i(t)$$

$y_i(t)$ = <u>the inner approximation</u>

11. Singular perturbation

When do regular perturbation fails?

E.g. if

1. When the highest order derivative is multiplied by ε.
2. When setting the parameter ε equal to zero completely changes the character of the problem.
3. When problems occur on infinite domains.
4. When singular points are present.
5. When the equation model physical processes that have multiple time or length scales.

1.–5. are singular perturbation problems.

In many cases we are concerned with problems involving boundary layers. Roughly speaking, we handle such problems as follows:

A. Let $\varepsilon = 0$ in the equation and we obtain often a good solution in the outer region.
B. The inner approximation in the bounder layer is found by rescaling.
C. The inner and outer approximations are matched.

$\therefore$ Singular perturbation = matched ~~asymptotic~~ expansions.

$$(*)\quad \begin{cases} \varepsilon y'' + (1+\varepsilon) y' + y = 0, \quad 0 < t < 1, \quad 0 < \varepsilon \ll 1, \\ y(0) = 0 \quad , \quad y(1) = 1 \end{cases}$$

12. The outer approximation

$$y' + y = 0 \quad , \quad y(1) = 1$$

$$\underline{y_0(t) = e^{1-t}} \qquad t = O(1)$$

13. The inner approximation

Scaling $\tau = \frac{t}{\delta(\varepsilon)}$ in $(*)$ gives

$$(**)\quad \frac{\varepsilon}{(\delta(\varepsilon))^2} z'' + \frac{1+\varepsilon}{\delta(\varepsilon)} z' + z = 0 ,$$

where $z(\tau) = y(\tau\,\delta(\varepsilon))$ $(= y(t))$.

Consider the coefficients

$$\frac{\varepsilon}{\delta(\varepsilon)^2}, \ \frac{1}{\delta(\varepsilon)}, \ \frac{\varepsilon}{\delta(\varepsilon)}, \ 1$$

In order to <u>**simplify**</u> the problem we choose the important term $\frac{\varepsilon}{\delta(\varepsilon)^2}$ of the same magnetude as <u>**one**</u> other coefficient and such that **the other two** are small in comparison.

(i) $\quad \frac{\varepsilon}{\delta(\varepsilon)^2} \approx \frac{1}{\delta(\varepsilon)} \qquad \frac{\varepsilon}{\delta(\varepsilon)^2} \gg \frac{\varepsilon}{\delta(\varepsilon)}, 1$

$\therefore \delta(\varepsilon) \approx \varepsilon$

(ii) $\quad \frac{\varepsilon}{\delta(\varepsilon)^2} \approx 1 \qquad \frac{\varepsilon}{\delta(\varepsilon)^2} \gg \frac{1}{\delta(\varepsilon)}, \frac{\varepsilon}{\delta(\varepsilon)}$

$\therefore \delta(\varepsilon) \approx \sqrt{\varepsilon} \qquad$ Impossible!

(iii) $\frac{\varepsilon}{\delta(\varepsilon)^2} \approx \frac{\varepsilon}{\delta(\varepsilon)}$, $\frac{\varepsilon}{\delta(\varepsilon)^2} >> \frac{1}{\delta(\varepsilon)}, 1$

$\therefore \delta(\varepsilon) \approx 1$ Impossible.

Choose $\delta(\varepsilon) = \varepsilon$ and $(**)$ becomes

$$z'' + z' + \varepsilon z' + \varepsilon z = 0$$

$\varepsilon = 0$ gives $z'' + z' = 0$ which has the solution

$$z = a + b e^{-\tau}$$

The boundary condition $z(0) = 0$ gives

$$z = a(1 - e^{-\tau}), \text{ i.e.,}$$

$$y(t) = a(1 - e^{-t/\varepsilon})$$

Summing up we obtain that

$$\underline{y_i(t)} = a(1 - e^{-t/\varepsilon}) \quad t = O(\varepsilon)$$

Our problem is now to determine the constant a and "match" the inner and outer approximations.

14. Matching

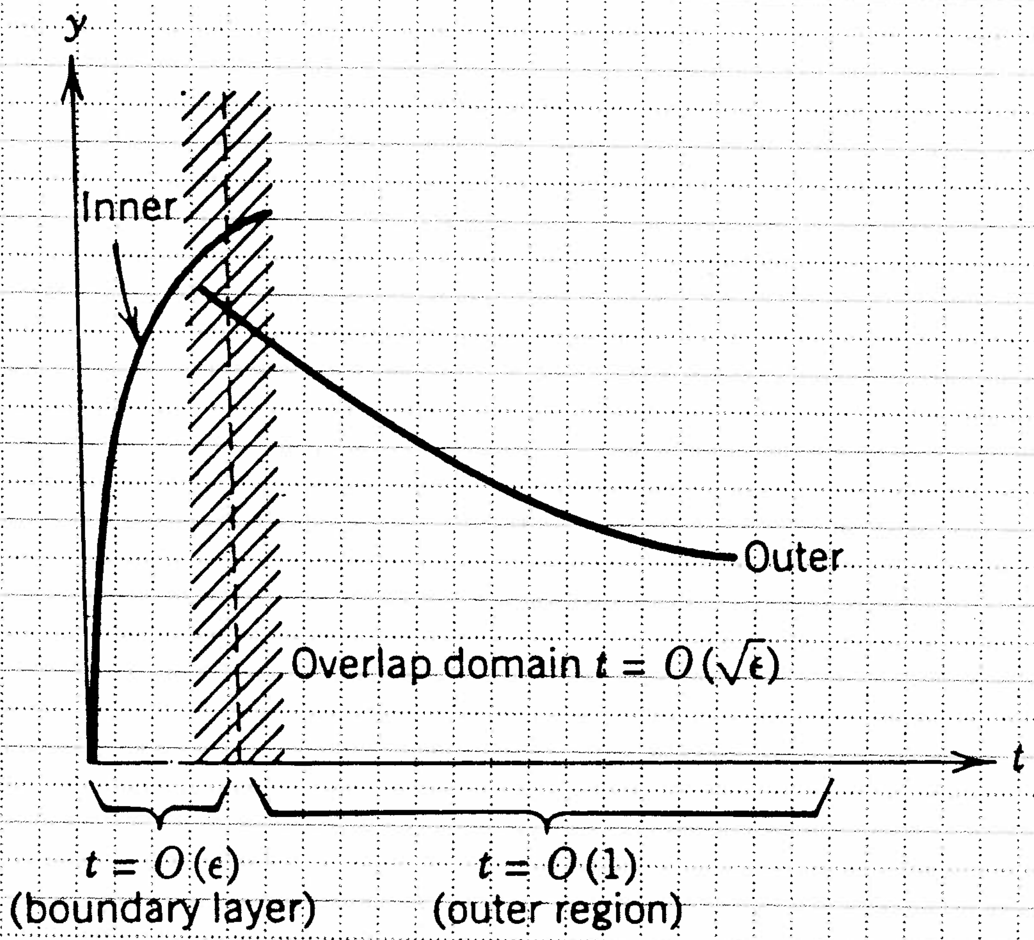


Figure 2.5. Overlap domain.

In overlap domain we let $t = O(\sqrt{\varepsilon})$ and introduce the <u>intermediate variable</u>

$$\eta = \frac{t}{\sqrt{\varepsilon}}$$

For <u>matching</u> we require that

$$\lim_{\varepsilon \to 0+} y_0(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} y_i(\sqrt{\varepsilon}\,\eta)$$

In our case

a) $\displaystyle \lim_{\varepsilon \to 0+} y_0(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} \exp(1 - \sqrt{\varepsilon}\,\eta) = e$

b) $\displaystyle \lim_{\varepsilon \to 0+} y_i(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} a(1 - \exp(-\eta/\sqrt{\varepsilon})) = a$

$\therefore$ $a = e$ and $y_i(t) = e(1 - e^{-t/\varepsilon})$.

15. A final example

Example: $\varepsilon y'' + y' = 2t$, $0 < t < 1$, $0 < \varepsilon << 1$,
$y(0) = 1$, $y(1) = 1$

A. "Outer approximation"

$\varepsilon = 0 \Rightarrow y' = 2t \Rightarrow y = t^2 + c$

$y(1) = 1 \Rightarrow c = 0$

$\therefore \underline{y_c(t) = t^2} \quad t = O(1)$

B. "Inner approximation"

Put $\tau = \frac{t}{\delta(\varepsilon)}$. Then

$$(**) \quad \frac{\varepsilon}{\delta(\varepsilon)^2} z'' + \frac{1}{\delta(\varepsilon)} z' - 2\delta(\varepsilon)\tau = 0$$

where $z(\tau) = y(\tau\delta(\varepsilon)) (= y(t))$

(i) $\frac{\varepsilon}{\delta(\varepsilon)^2} \approx 2\delta(\varepsilon) \quad \frac{\varepsilon}{\delta(\varepsilon)^2} >> \frac{1}{\delta(\varepsilon)}$

$\therefore \delta(\varepsilon) \approx \varepsilon^{1/3}$ Impossible

(ii) $\frac{\varepsilon}{\delta(\varepsilon)^2} \approx \frac{1}{\delta(\varepsilon)} \quad \frac{\varepsilon}{\delta(\varepsilon)^2} >> 2\delta(\varepsilon)$

$\therefore \delta(\varepsilon) \approx \varepsilon$

Choose $\delta(\varepsilon) = \varepsilon$ in $(**)$:

$$z'' + z' = 2\varepsilon^2\tau$$

The inner approximation to first order

$$z'' + z' = 0$$

$$z(\tau) = a + b e^{-\tau} \text{ i.e. } y(t) = a + b e^{-t/\varepsilon}$$

The condition $y(0)=1$ gives that

$$\underline{y_i(t)} = (1-b) + b\,e^{-t/\varepsilon}$$

C. "Matching"

Introduce the intermediate variable

$$\eta = t/\sqrt{\varepsilon}$$

Mathing condition

$$\lim_{\varepsilon \to 0+} y_0(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} y_i(\sqrt{\varepsilon}\,\eta)$$

gives that

$$\lim_{\varepsilon \to 0+} \varepsilon\eta^2 = \lim_{\varepsilon \to 0+} \{(1-b) + b\,e^{-\eta/\sqrt{\varepsilon}}\}, \text{ i.e.,}$$

$$0 = 1-b \iff b = 1$$

$$\therefore \quad \underline{y_i(t) = e^{-t/\varepsilon}}$$

16. The WKB approximation

The WKB-method (Wentzel Kramers-Brillouin) is a perturbation method which applies to several problems e.g. the following type:

$$\begin{cases} \varepsilon^2 y'' + q(x)\, y = 0, & 0 < \varepsilon << 1 \\ y'' + (\lambda^2 p(x) - q(x))\, y = 0, & \lambda >> 1 \\ y'' + q(\varepsilon x)^2 y = 0 & \end{cases}$$

The time-independent Schrödinger equation

$$-\frac{h^2}{2m} y'' + (V(x) - E)\, y = 0$$

appears in a natural way. This is one of the most fundamental equations in mathematical physics. etc.

Problems - Lecture 2

1. Consider the initial value problem

$$u'' - u = \varepsilon t u\ ,\quad t>0;\quad u(0)=1,\ u'(0)=-1.$$

Fin a two-term perturbation approximation for $0<\varepsilon \ll 1$ and compare it graphically with a six-term Taylor series approximation (centered at $t=0$) where $\varepsilon = 0.04$. Use a numerical differential equation solver (e.g. with Maple) to find the exact solution and compare.

2. Verify the following order relations:

a) $t^2 \tanh t = O(t^2)$ as $t \to \infty$,

b) $\exp(-t) = o(1)$ as $t \to \infty$,

c) $\sqrt{\varepsilon(1-\varepsilon)} = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0+$,

d) $\dfrac{\sqrt{\varepsilon}}{1-\cos\varepsilon} = O(\varepsilon^{-3/2})$ as $\varepsilon \to 0+$,

e) $\exp\varepsilon - 1 = O(\varepsilon)$ as $\varepsilon \to 0$.

3.* Use the Poincaré-Lindstedt method to obtain a two-term perturbation approximation to the initial value problem:

$$y'' + y = \varepsilon y (y')^2,\quad y(0)=1,\ y'(0)=0.$$

4.* Consider the initial value problem
$y'' = \varepsilon t y$, $0 < \varepsilon << 1$, $y(0) = 0$, $y'(0) = 1$.
Using regular perturbation theory obtain a three-term approximation solution on $t \geq 0$.

5. Show that regular perturbation fails on the boundary value problem

$$\varepsilon y'' + y' + y = 0, \; 0 < t < 1, \; 0 < \varepsilon << 1,$$
$$y(0) = 0, \; y(1) = 1.$$

Find the exact solution for $\varepsilon = 0.05$ and $\varepsilon = 0.005$. If $t = O(\varepsilon)$, then show that $\varepsilon y''(t)$ is large; if $t = O(1)$, then show that $\varepsilon y''(1) = O(1)$. Find the inner and the outer approximations from the exact solution

6.* Use singular perturbation methods to obtain a uniform approximate solution to the following problem:

$$\begin{cases} \varepsilon y'' + 2y' + y = 0, \; 0 \leq t \leq 1, \\ y(0) = 0, \; y(1) = 1, \; 0 < \varepsilon << 1. \end{cases}$$

7. By examining the exact solution show why singular perturbation methods fail on the boundary value problem:

$$\begin{cases} \varepsilon y'' + y = 0, \; 0 < t < 1, \; 0 < \varepsilon << 1, \\ y(0) = 1, \; y(1) = 2. \end{cases}$$