III CALCULUS OF VARIATIONS

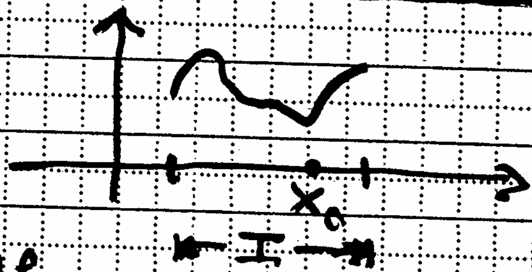
1. Functions – stationary or extremepoints

Consider a real-valued function

$$f: I \to R,$$

where I is an open interval on R. We say that f has <u>a local minimum</u> at $x_0 \in I$ if

$$f(x_0) \leq f(x)$$



for all x in some neighbourhood of x_0 in I. The definition of <u>a local maximum</u> is similar. It is well-known that if f is differentiable, then a necessary condition that x_0 is an extremepoint (a local minimum or a local maximum) is that

$$(*) \qquad f'(x_0) = 0$$

The converse is **not** true but, if in addition,

$$f''(x_0) > 0 \quad (f''(x_0) < 0)$$

then f has a local minimum (a local maximum) at x_0.

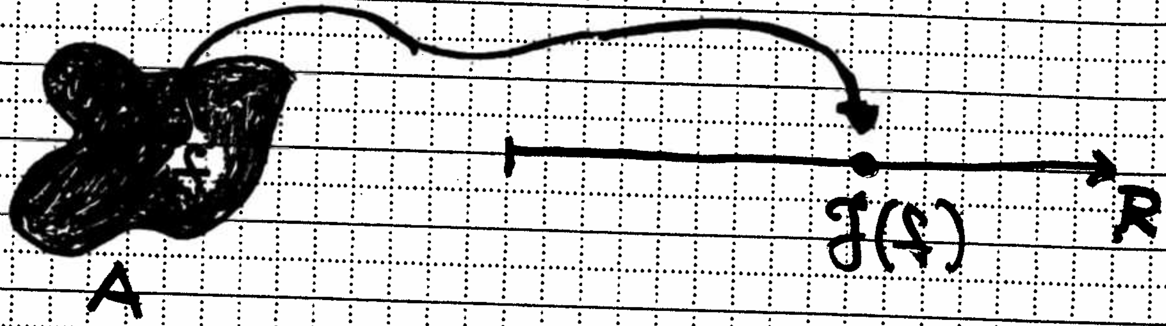
<u>Remark:</u> A point x_0 satisfying $(*)$ is called a <u>stationary point</u> of f.

2. Functionals - stationary or extreme "vectors" (functions)

Consider a real-valued functional

$$J : A \rightarrow R$$

where A is a set of "admissible" functions.



Fundamental problem of calculus of variations: Given a functional J and an admissible set of functions, determine which functions in A which give an extreme value to J.

Remark: In most cases J are defined by some integral and the functions are functions in the classical sense.

Remark: Our fundamental problem to extremizing a functional J over a set A is (sometimes) called a variational problem.

3. Some function spaces

Example 1: $C[a,b]$ is the set of continuous functions $y-y(a)$ on $[a,b]$.
As a norm we can choose

$$\|y\|_\infty = \max_{a\le x\le b} |y(x)|,$$

$$\|y\|_1 = \int_a^b |y(x)|\,dx$$

$$\|y\|_2 = \left(\int_a^b |y(x)|^2\,dx\right)^{1/2}$$

Example 2: $C^1[a,b]$ is the set of all continuous functions on $[a,b]$ such that the derivative is also continuous.
As a norm we can choose the weak norm given by

$$\|y\|_W = \max_{a\le x\le b} |y(x)| + \max_{a\le x\le b} |y'(x)|$$

or

$$\|y\| = \|y\|_2 + \|y'\|_2 .$$

Example 3: The spaces $C^n[a,b]$, $n=2,3,\dots$ are defined in a similar way.

Distance between two functions y_1, y_2:

$$\|y_1 - y_2\|.$$

Examples 1-3 are examples of Normed

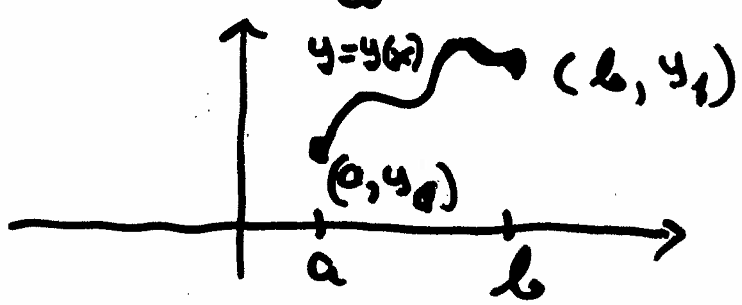
4. Some examples of variational problems.

Example 4: $A = \{ f \in C^1[a,b],\ y(a) = y_0,\ y(b) = y_1 \}$

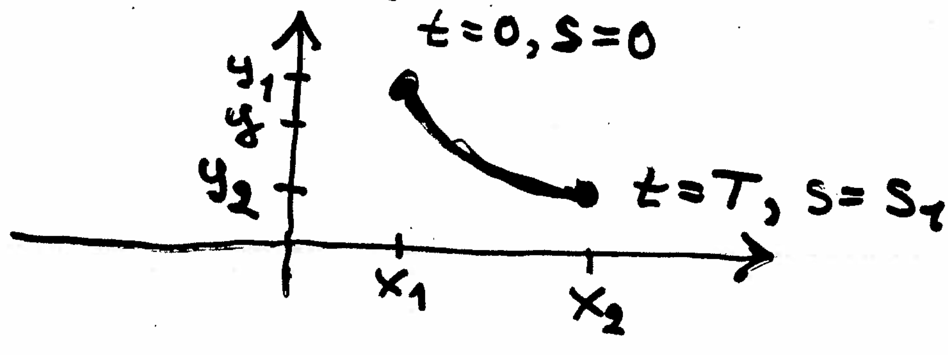
Find $\min J(y)$ when $y \in A$ and

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2}\,dx .$$

the arc length



Example 5 (The Brachistochrone Problem): A bead of mass m with initial velocity zero slides with no friction under the force of gravity g from a point (x_1, y_1) to a point (x_2, y_2) along a wire defined by a curve $y = y(x)$ in the xy-plane. Which curve leads to the fastest time of descent? (Bernoulli 1696?)



This problem leads to the variational problem to minimize

$$T(y) = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g\,(y_1 - y(x))}}\,dx$$

when $A = \{ y \in C^1[$

Example 6: Minimize

$$J(y) = \int_{x_1}^{x_2} y(x)\,dx$$

when $A = \{ y \in C[a,b],\ y(x) \neq 0 \}$

Interpretation and answer ?

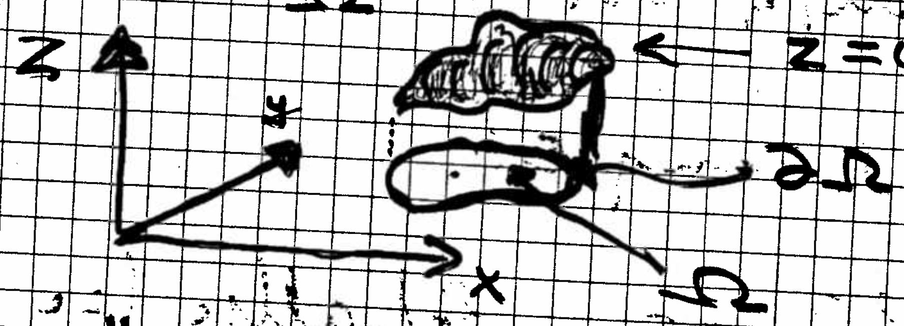
Example 7: Let A be any set of functions on $[a,b]$. Then

$$J(y) = y(x_0) \quad , \; x_0 \in [a,b],$$

is a functional.

Example 8: "Minimum surface problem"

Find $\min \{ F(u) = \iint_{\Omega} \sqrt{1+(u'_x)^2+(u'_y)^2}\,dx : u = u_0 \text{ on } \partial\Omega \}$



Example 9: "Dirichlet's integral" (≈ The principle of minimum energy)

Find $\min \{ F(u) = \iint_{\Omega} |\nabla u|^2 dx \, ; u = u_0 \text{ on } \partial\Omega \}$.

$$\left(\nabla u = (u'_x, u'_y) \quad |\nabla u|^2 = (u'_x)^2 + (u'_y)^2 \right)$$

5. The Euler equation for the Simplest Problem

Theorem 1: If a function y provides a local extremale to the functional

$$J(y) = \int_a^b L(x, y, y')\,dx,$$

where $y \in C^2[a,b]$ and

$$y(a) = y_0, \quad y(b) = y_1,$$

then y must satisfy the equation

$$(*) \qquad L_y(x,y,y') - \frac{d}{dx} L_{y'}(x,y,y') = 0, \quad x \in [a,b]$$

Remark 1: $(*)$ is called the Euler equation. It corresponds to the condition $f'(x_0) = 0$ for the function case.

Remark 2: $(*)$ is a second order ODE that is in general nonlinear. It can be written as

$$L_y - L_{y'x} - L_{y'y} \cdot y' - L_{y'y'}\, y'' = 0.$$

Remark 3: The function $L = L(x,y,y')$ is called the Lagrangian of the problem. In our examples 4, 5, 6 we had

$$L = \sqrt{1 + (y')^2},$$

$$L = \sqrt{1 + (y')^2} / \sqrt{2g(y_1 - y)},$$

and

$$L = y,$$

Example 10: Find the extremals of the functional

$$J(y)=\int_0^1\left((y')^2+3y+2x\right)dx\ ,\ y(0)=y(1)=1.$$

Sol: Here we have

$$L=(y')^2+3y+2x.$$

Then $L_y=3$ and $L_{y'}=2y'$ and Euler's equation becomes

$$3-\frac{d}{dx}(2y')=0 \Leftrightarrow y''=\frac{3}{2}.$$

Thus

$$y(x)=\frac{3}{4}x^2+ax+b.$$

Hence, according to the boundary conditions,

$$y(x)=\frac{3}{4}x^2-\frac{3}{4}x+1$$

Example 11: (c.f. Example 4) Find the extremal of the functional

$$J(y)=\int_a^b\sqrt{1+(y')^2}\,dx\ ,\ y(a)=y_0\ ,\ y(b)=y_1.$$

Sol: The Lagrangian is

$$L=\sqrt{1+(y')^2}$$

and, thus, by the Euler equation,

$$0-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)=0 \Leftrightarrow \frac{y'}{\sqrt{1+(y')^2}}=c.$$

Therefore $y'=K\ (K=c/\sqrt{1-c^2}) \Rightarrow y=Kx+l$

$$\therefore\ y=\frac{y_1-y_0}{b-a}x+\frac{by_0-ay_1}{b-a}.$$

7. Some "simplifications" of Euler's equation

1. $L = L(x,y)$. Then Euler's equation is
$$L_y(x,y) = 0.$$

2. $L = L(x,y')$. Then Euler's equation is
$$L_{y'}(x,y') = C.$$

3. $L = L(y,y')$. Then Euler's equation is
$$L(y,y') - y' L_{y'}(y,y') = C$$

Proof: $\frac{d}{dx}\left(L - y' L_{y'}\right) = \frac{dL}{dx} - y' \frac{d}{dx}(L_{y'}) - L_{y'} y'' =$

$$L_y \cdot y' + L_{y'} y'' - y' \frac{d}{dx}(L_{y'}) - L_{y'} y'' =$$

$$y'\left(L_y - \frac{d}{dx}(L_{y'})\right) = \{\text{Euler}\} = y' \cdot 0 = 0$$

Example 12: (c.f. Example 5) Find the extremal of the functional
$$J(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2g(y_1-y)}}\, dx, \quad y(x_1) = y_1, \; y(x_2) = y_2.$$

Sol: Here we have
$$L = L(y,y') = \sqrt{1+(y')^2} \Big/ \sqrt{2g(y_1-y)}$$
and we get simpler calculations by using **3** above. We get

$$\frac{\sqrt{1+(y')^2}}{\sqrt{y_1-y}} - (y')^2 \frac{(1+(y')^2)^{-1/2}}{\sqrt{y_1-y}} = C,$$

which can be simplified to

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-C^2(y_1-y)}{C^2(y_1-y)}.$$

Hence

$$dx = -\frac{\sqrt{y_1-y}}{\sqrt{C_1-(y_1-y)}}\,dy \quad (C_1 = C^{-2}).$$

(We take the minus sign because $\frac{dy}{dx} < 0$).

Put

(1) $\quad y_1 - y = C_1 \sin^2\frac{\theta}{2} \qquad \frac{dy}{d\theta} = -C_1 \sin\frac{\theta}{2}\cos\frac{\theta}{2}$

and one finds that

(2) $\quad dx = C_1 \sin^2\frac{\theta}{2}\,d\theta = \frac{C_1}{2}(1-\cos\theta)\,d\theta$

(1) and (2) gives

$$(*)\quad \begin{cases} x = \dfrac{C_1}{2}(\theta - \sin\theta) + C_2 \\[2mm] y = y_1 - \dfrac{C_1}{2}(1-\cos\theta) \end{cases}$$

The constants C_1 and C_2 are calculated by using the boundary conditions

(*) is a <u>cykloid</u> on parameter form.

8. Proof of Theorem 1

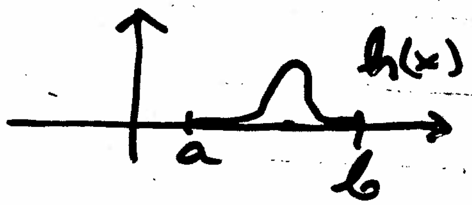
For the proof we need the following

Lemma Let $f \in C[a,b]$ and assume that

$$\int_a^b f(x)h(x)\,dx = 0$$

for every $h \in C^2[a,b]$, $h(a) = h(b) = 0$. Then $f(x) \equiv 0$.

Proof: See the book.



"Proof" of Theorem 1: Let y be an extremal and let $h \in C^2[a,b]$, $h(a) = h(b) = 0$. Then $y + \varepsilon h$ is an admissible and competing function for $\varepsilon > 0$.

$$J(y+\varepsilon h) = \int_a^b L(x, y+\varepsilon h, y' + \varepsilon h')\,dx.$$

Since y is extremal we must have

$$(1) \qquad \left\{\frac{d}{d\varepsilon} J(y+\varepsilon h)\right\}_{\varepsilon=0} = 0.$$

We have

$$\frac{d}{d\varepsilon} J(y+\varepsilon h) = \int_a^b \frac{d}{d\varepsilon}\left(L(x, y+\varepsilon h, y'+\varepsilon h')\right)dx =$$

$$= \int_a^b \{L_y(x, y+\varepsilon h, y'+\varepsilon h')h + L_{y'}(x, y+\varepsilon h, y'+\varepsilon h')h'\}dx$$

According to (1) we conclude that $=0$

$$(2) \quad \int_a^b \{L_y(x,y,y')h + L_{y'}(x,y,y')h'\}\,dx = 0$$

for all $h \in C^2[a,b]$ with $h(a)=h(b)=0$.

We integrate the second term in (2) by parts and find that

$$(3) \quad \int_a^b L_{y'}(x,y,y')h'\,dx = \underbrace{\left[L_{y'}\cdot h\right]_{x=a}^{x=b}}_{=0} - \int_a^b \frac{d}{dx}\big(L_{y'}(x,y,y')\big)h\,d$$

Thus, by (2) and (3), we have

$$\int_a^b \Big(L_y(x,y,y') - \frac{d}{dx}\{L_{y'}(x,y,y')\}\Big)h\,dx = 0$$

for all $h \in C^2[a,b]$ with $h(a)=h(b)=0$.

Finally we use the Lemma and conclude that

$$L_y(x,y,y') - \frac{d}{dx}\{L_{y'}(x,y,y')\} = 0.$$

The proof is complete.

9. Natural Boundary Conditions

Assume that we in Theorem 1 instead have the boundary conditions

$$y(a) = y_0 \; , \; y(b) \text{ unspecified.}$$

By following the proof of Theorem 1 we find that we must have

$$L_{y'}(b, y(b), y'(b)) \cdot h(b) = 0$$

for all choices of h satisfying $h(a) = 0$. This means that

$$L_{y'}(b, y(b), y'(b)) = 0 .$$

This is the natural Boundary condition.

Example 13: Find the differential equation and the boundary conditions for the extremal of the functional

$$J(y) = \int_0^1 (x^2 (y')^2 - x^3 y^2) \, dx$$

$$y(0) = 0 \; , \; y(1) \text{ unspecified.}$$

Sol $L_y = -2x^3 y \quad L_{y'} = 2x^2 y' \quad L = x^2 (y')^2 - x^3 y^2$

The Euler equation is

$$-2x^3 y - \frac{d}{dx} 2x^2 y' = 0 \text{ i.e.}$$

$$x^2 y'' + 2x y' + x^3 y = 0$$

The natural boundary condition at $x = 1$ is

$$2 \cdot 1^2 y'(1) = 0$$

The boundary conditions are $y(0) = 0$ and $y'(1) = 0$

10. The Euler equation in some more general cases.

I. Higher derivatives

$$J(y)=\int_a^b L(x,y,y',y'')\,dx \quad , y\in C^4[a,b]$$

$y(a)=A_1,\ y'(a)=A_2,\ y(b)=A_3,\ y'(b)=A_4.$

$$\boxed{L_y-\frac{d}{dx}(L_{y'})+\frac{d^2}{dx^2}(L_{y''})=0}$$

II. Several functions

$$J(y_1,y_2)=\int_a^b L(x,y_1,y_2,y_1',y_2')\,dx$$

$y_1(a)=A_1,\ y_2(a)=A_2,\ y_1(b)=A_3,\ y_2(b)=A_4.$

$$\boxed{\begin{cases} L_{y_1}-\frac{d}{dx}(L_{y_1'})=0\\ L_{y_2}-\frac{d}{dx}(L_{y_2'})=0\end{cases}}$$

III. Multiple Integral Problems

$$J(u)=\iint_\Omega L(x,y,u(x,y),u_x(x,y),u_y(x,y))\,dx\,dy$$

$$\boxed{L_u-\frac{\partial}{\partial x}(L_{u_x})-\frac{\partial}{\partial y}(L_{u_y})=0}$$

Example 14 (c.f. Example 9) The Euler equation of $J(u)=\iint_\Omega |\nabla u|^2\,dx\,dy$ is

$$\Delta u=u''_{xx}+u''_{yy}=0 \text{ in } \Omega. \qquad \boxed{\text{Laplace equation}}$$

Example 15: (see Example 8) The Euler equation of $J(u) = \iint_\Omega \sqrt{1+|\nabla u|^2}\,dxdy$ is

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\frac{\partial u}{\partial x_2}\right) = 0.$$

Example 16: ("överkurs") The Euler equation of $J(u) = \frac{1}{p}\iint_\Omega |\nabla u|^p\,dxdy$ is

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \qquad 1<p<\infty$$

"p-harmonic equation" "p-Laplacian"

(Non-Newtonian fluids)

11. Normed Linear Spaces

A collection V of objects (functions, vectors) is called a real linear space if

a) there exists an operation $+$ on V such that if $u \in V$, $v \in V$, then $u+v \in V$,

b) $u+v = v+u$ (commutativity),

c) $u+(v+w) = (u+v)+w$ (associativity),

d) there exists a zero element 0 in V such that $u+0 = u$ for all $u \in V$,

e) for each $u \in V$ there exists an inverse element denoted $-u$ such that $u+(-u) = 0$,

f) for all $u \in V$ and $\alpha \in R$ the scalar multiplication αu is defined and $\alpha u \in V$,

g) $\alpha(\beta u) = (\alpha\beta)u$,
$(\alpha+\beta)u = \alpha u + \beta u$,
$\alpha(u+v) = \alpha u + \alpha v$,
for all $\alpha, \beta \in R$ and all $u, v \in V$.

h) $1u = u$.

Example 17: The set R^n.

objects?: $(x_1, x_2, \dots, x_n)$, $x_n \in R$.

$+$?: $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$

αx?: $\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$.

0?: $(0, 0, \dots, 0)$.

$-x$?: $(-x_1, -x_2, \dots, -x_n)$.

Example 18: The set $C[a,b]$

objects : Continuous functions on $a \le x \le b$.

$+$: $(f+g)(x) = f(x) + g(x)$

αf : $\alpha f(x) = \alpha f(x)$

0 : $f \equiv 0$ on $a \le x \le b$

$-f$: $-f(x)$ on $a \le x \le b$

Example 19: $C^n[a,b]$

Example 20: The set V of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 0$ is <u>not</u> a linear space.

<u>Proof</u> Choose $v_1 = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

Then

$v_1 \in V$ (because $4\cdot 4 - 2\cdot 8 = 0$)

$v_2 \in V$ (because $2\cdot 6 - 4\cdot 3 = 0$)

but

$$v_1 + v_2 = \begin{pmatrix} 10 & 11 \\ 6 & 6 \end{pmatrix} \notin V \text{ (because } 10\cdot 6 - 6\cdot 11 \neq 0\text{)}.$$

We conclude that a) fails and the proof is complete.

A <u>normed linear space</u> is a linear space equipped with a <u>norm</u> that is a mapping that associates to each $y \in V$ a real number denoted by $\|y\|$ satisfying

a) $\|y\| = 0 \Leftrightarrow y \equiv 0$,

b) $\|\alpha y\| = |\alpha| \, \|y\|$ for all $\alpha \in \mathbb{R}$, $y \in V$,

c) $\|y_1 + y_2\| \le \|y_1\| + \|y_2\|$ for all $y_1, y_2 \in V$.

<u>Example 21</u>: $\mathbb{R}^n$ equipped with one of the norms

$$\|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2},$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|,$$

or

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|.$$

<u>Example 22</u> $C[a,b]$ equipped with one of the norms

$$\|y\|_\infty = \max_{a \le x \le b} |y(x)|,$$

$$\|y\|_2 = \left(\int_a^b |y(x)|^2 dx\right)^{1/2},$$

or

$$\|y\|_1 = \int_a^b |y(x)| dx.$$

"<u>Distance</u>" between y_1 and y_2 : $\|y_1 - y_2\|$

Thus, a norm always induces a corresponding way to measure "<u>distance</u>" between the objects.

12. Local minimum of a functional

Let $\mathcal{J} : X \to \mathbb{R}$ be a functional, where X is a normed linear space with norm $\|\cdot\|$.

We say that $\mathcal{J}$ has a local minimum at $y_0 \in X$ if

$$\mathcal{J}(y_0) \le \mathcal{J}(y)$$

for all $y \in X$ with $\|y - y_0\| < \delta$ for some $\delta > 0$.

Remark; Local maximum is defined in an obvious similar way.

13. Differentiation of a functional

For $u, v \in X$ (X is a vector space) we define the Gâteau variation of $\mathcal{J}$ in the direction v by

$$\delta \mathcal{J}(u; v) = \lim_{\varepsilon \to 0} \frac{\mathcal{J}(u + \varepsilon v) - \mathcal{J}(u)}{\varepsilon}$$

Provided this limit exists.

Remark: $\lim_{\varepsilon \to 0} \frac{\mathcal{J}(u+\varepsilon v) - \mathcal{J}(u)}{\varepsilon} = \left\{ \frac{d}{d\varepsilon} \mathcal{J}(y_0 + \varepsilon h) \right\}_{\varepsilon = 0}$

Why ?

Example 23: $J = f \in C^1[a,b]$, $x, y \in X = R^n$

$$\delta f(x;y) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon y) - f(x)}{\varepsilon},$$

i.e. the directional derivative in the direction y when y is a unit vector.

By making Taylor expansion we see that

$$\delta f(x;y) = \nabla f(x) \cdot y$$

Example 24: (c.f. Theorem 1 and the proof in section 8)

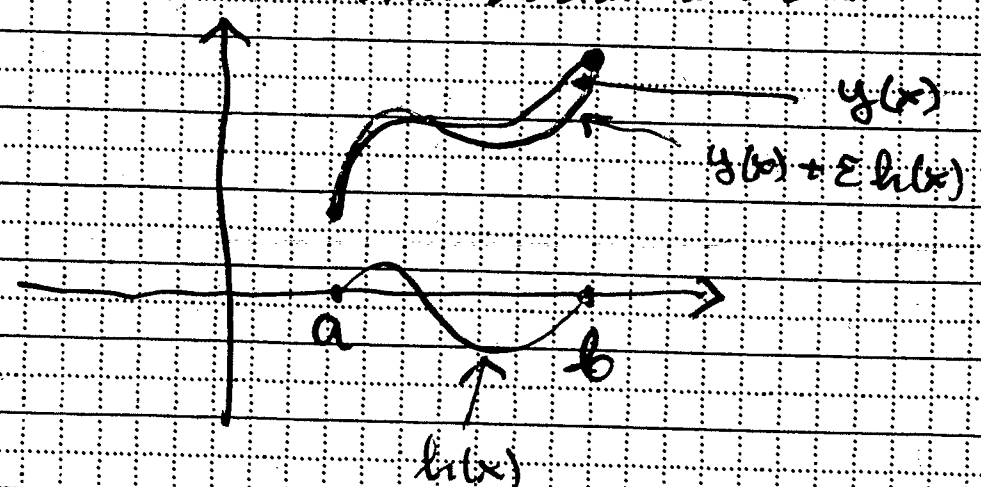
$$F(u) = \int_a^b L(x, y, y')\,dx, \quad y, h \in C^2[a,b$$

Then

$$\delta f(y;h) = \left\{\frac{d}{d\varepsilon} J(y+\varepsilon h)\right\}_{\varepsilon=0} =$$

$$= \int_a^b \left(L_y(x,y,y') - \frac{d}{dx}\left(L_{y'}(x,y,y')\right)\right) h\,dx + \Big[L_{y'} h\Big]_{x=(}^{x=b}$$

Geometric intepretation



14. A necessary condition for minimum (extremum) of a functional J

Theorem 2: Let $J: A \to R$ be a given functional, $A \subset V$, where V is a normed linear space. If $y_0 \in A$ provides a local minimum for J relative to the norm $\|\cdot\|$, then

$$(*) \qquad \delta J(y_0, h) = 0$$

for all admissible variations h.

Remark: The solutions of $(*)$ are usually called extremals.

Proof: Define the function

$$\varphi(\varepsilon) = J(y_0 + \varepsilon h).$$

Obviously $\varphi: R \to R$ is a usual one-variable function and extremum occur for $\varepsilon = 0$ exactly if $\varphi'(0) = 0$ i.e. exactly if

$$\left(\frac{d}{d\varepsilon} J(u_0 + \varepsilon u)\right)_{\varepsilon=0} = \delta J(y_0, h) = 0.$$

The proof is complete.

Problems - Lecture 3

1.* Consider the functional

$$J(y) = \int_a^b L(x, y, y')\,dx,$$

where $y \in C^2[a,b]$, $y(a) = y_0$ and $y(b) = y_1$.

a) Derive the Euler equation for this functional.

b) Derive the natural boundary condition at the right endpoint when the condition "$y(b) = y_1$" is removed.

2. A functional $J: V \to R^1$, where V is a normed linear space, is linear if

$$J(y_1 + y_2) = J(y_1) + J(y_2),\ y_1, y_2 \in V$$

and

$$J(\alpha y_1) = \alpha J(y_1),\ \alpha \in R,\ y_1 \in V.$$

Which of the following functionals on $C^1[a,b]$ are linear?

a) $J(y) = \int_a^b y\,y'\,dx$, b) $J(y) = \int_a^b ((y')^2 + 2y)\,dx$,

c) $J(y) = \int_a^b y \sin x\,dx$, d) $J(y) = \int_a^b \int_a^b e^{x^2+t^2} y(x)\,y(t)\,dx\,dt$

3. Show that the Euler equation for the functional

$$J(y) = \int_a^b f(x,y)\sqrt{1+(y')^2}\,dx$$

has the form:

$$f'_y - f'_x y' - f y''/(1+(y')^2) = 0.$$

4*. Find an extremal for

$$J(y)=\int_1^2 \frac{\sqrt{1+(y')^2}}{x}\,dx\ ,\ y(1)=0\ ,\ y(2)=1.$$

5. Find the extremals for

$$J(y)=\int_a^b \left(x^2(y')^2+y^2\right)dx\ ,\ y\in C^2[a,b].$$

6. Find the extremals for

a) $J(y)=\int_0^1 \left(yy'+(y'')^2\right)dx\ ,\ y(0)=0,\ y'(0)=1,\ y(1)=2,\ y'(1)=4$

b) $J(y)=\int_0^\infty \left(y^2+(y')^2+(y''+y')^2\right)dx,$

$y(0)=1,\ y'(0)=2,\ y(\infty)=0,\ y'(\infty)=0.$

7*. Find an extremal for

$$J(y)=\int_1^e \left(\frac{1}{2}x^2(y')^2-\frac{1}{8}y^2\right)dx\ ,\ y(1)=1,\ y(e)\ \text{unspecified}$$