LECTURE 4

Introduction to PDE

1. Some examples

Ex 1: (One-dimensional heat equation)

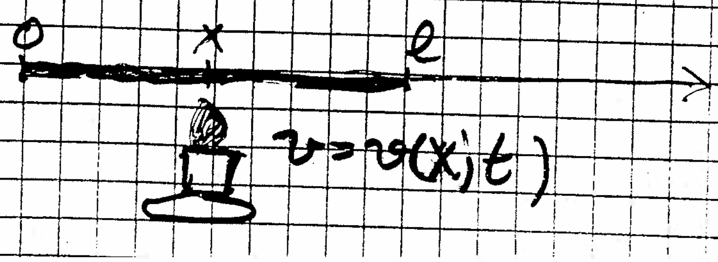
$u(x,t)$ = heat in the point x at the time t.

$$
\begin{aligned}
&u'_t - k u''_{xx} = 0 &&, t>0, 0<x<l\\
&u(x,0) = f(x) &&, 0<x<l,\\
&u(0,t) = h(t) &&, t>0,\\
&u(l,t) = g(t) &&, t>0.
\end{aligned}
$$

Ex 2: (One-dimensional heat equation - "nonhomogeneous" case)

$$
\begin{aligned}
&u'_t - k u''_{xx} = v &&, t>0, 0<x<l\\
&u(x,0) = f(x) &&, 0<x<l,\\
&u(0,t) = h(t) &&, t>0,\\
&u(l,t) = g(t) &&, t>0.
\end{aligned}
$$

$v = v(x,t)$ = added heat in the point x at the time t.

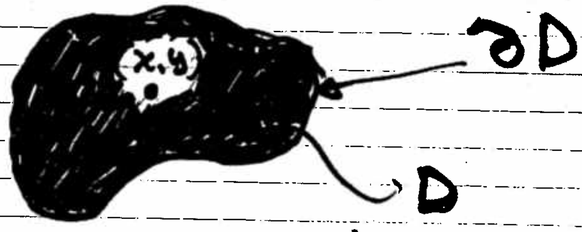


EX3: (Two-dimensional heat conduction)

$u = u(x,y,t)$ = heat in the point (x,y) at the time t.

$$u'_t - \cancel{k}(u''_{xx} + u''_{yy}) = v \,, (x,y) \in D, t > 0$$

$$u(x,y,0) = f(x,y) \,, (x,y) \in D$$
$$u(x,y,t) = g(x,y) \,, (x,y) \in \partial D, t > 0$$



$v(x,y,t)$ = added heat in the point (x,y) at the time t.

EX4: (Three-dimensional heat conduction)

$u = u(x,y,z,t)$, $v = v(x,y,z,t)$

(*) $$u'_t - \operatorname{div}(k \operatorname{grad} u) = v \,, t > 0, (x,y,z) \in V$$
$$u(x,y,z,0) = f(x,y,z) \,, (x,y,z) \in V$$

$$u(x,y,z,t) = g(x,y,z) \,, (x,y,z) \in \partial V, t > 0$$

Remark: If $k = k(x,y,z)$ is constant $= k_0$ then (*) can be written

$$u'_t - k_0(u''_{xx} + u''_{yy} + u''_{zz}) = v$$

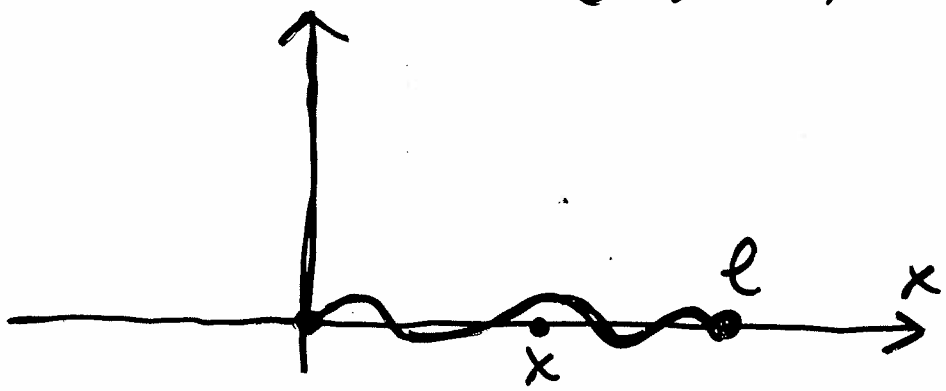
$\Leftrightarrow$

$$u'_t - k_0 \Delta u = v \qquad \boxed{\nabla \cdot \nabla = \Delta}$$

$\nabla u = (u'_x, u'_y, u'_z) = \operatorname{grad} u \ldots$

EX5: (One-dimensional wave equation)

$$u''_{tt} - k\, u''_{xx} = 0, \quad 0<x<\ell, \ t\geq 0,$$
$$u(0,t) = u(\ell,t) = 0, \quad t\geq 0,$$
$$u(x,0) = f(x), \quad 0<x<\ell,$$
$$u'_t(x,0) = g(x), \quad 0<x<\ell.$$



$u(x,t)$ = position of the vibrating string at the time t

EX6: Two-dimensional wave equation

$$u''_{tt} - k(u''_{xx} + u''_{yy}) = 0, \quad (x,y)\in D, \ t>0,$$
$$u(x,y,t) = 0, \quad (x,y)\in \partial D, \ t\geq 0,$$
$$u(x,y,0) = f(x,y), \quad (x,y)\in \partial D,$$
$$u'_t(x,y,0) = g(x,y), \quad (x,y)\in \partial D.$$

"Shape of a vibrating membrane".

Ex 7: (Two-dimensional Laplace equation)

(*) $$u''_{xx} + u''_{yy} = 0 \Leftrightarrow \nabla^2 u = 0$$

(let $u'_t = 0$ and $v = 0$ in ex 3! Thus (*) can e.g. mean that $u = u(x,y)$ is the heat at the point (x,y) when we have reached equilibrium)

Ex 8: (Two-dimensional Poisson equation

$$u''_{xx} + u''_{yy} = f \Leftrightarrow \nabla^2 u = f$$

Physical interpretation?

Ex 9: (Three dimensional Poisson equatio

$$u''_{xx} + u''_{yy} + u''_{zz} = f \Leftrightarrow \nabla^2 u = f$$

Physical interpretation?

2. A general PDE of second order

(1) $G(x, t, u, u'_x, u'_t, u''_{xx}, u''_{xt}, u''_{tt}) = 0$

BASIC QUESTIONS:

1. Does there exist a solution of (1)?
2. Is the solution unique?
3. Is the solutions stable for small perturbations?
4. Methods for constructing and illustrating solutions?

EX 10: The problems in examples 1-6 have unique solutions.

The problems in examples 7-9 have not unique solutions.

Remark: A PDE of the type (1) usually has infinitely many solutions. The general solution usually depends on arbitrary functions.

EX 11: The equation $u''_{tx} = tx$ has the solutions

$$u = \frac{1}{4}t^2x^2 + g(t) + h(x).$$

EX 12: Two dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0$$

has e.g. the (very different) solutions

$$u = x^2 - y^2, \quad u = e^x \cos y \text{ and } u = \ln(x^2 + y^2)$$

3. Linearity - Nonlinearity

A partial differential equation can be written on the form

$$(*) \qquad Lu = f$$

where L is a <u>differential operator</u>.

Ex 13: Let $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$. Then $(*)$ reads

$$u'_t - k u''_{xx} = f,$$

i.e. $(*)$ coincides with a one-dimensional heat equation (see Ex 2)

Ex 14: Let $L(u) = u\frac{\partial u}{\partial t} + 2txu$. Then $(*)$ reads

$$u\frac{\partial u}{\partial t} + 2txu = f(x,t).$$

Def: We say that $(*)$ is <u>linear</u> if L has the properties

$$(1) \qquad L(u+v) = Lu + Lv$$

$$(2) \qquad L(cu) = cLu$$

If some of these properties is not satisfied then we say that $(*)$ is <u>non-linear</u>.

Ex 15: The heat equation in Ex 13 is linear

Proof: (1) $L(u+v) = (u+v)'_t - k(u+v)''_{xx} = u'_t + v'_t - k u''_{xx} - k v''_{xx} = Lu + Lv$,

(2) $L(cu) = (cu)'_t - k(cu)''_{xx} = cu'_t - kcu''_{xx} = c(u'_t - ku''_{xx}) = cLu$.

Ex 16: The PDE in Ex 14 is non-linear.

Proof: $L(u+v) = (u+v)(u+v)'_t + 2tx(u+v) =$

$$\underline{uu'_t} + uv'_t + vu'_t + \underline{vv'_t} + \underline{2txu} + \underline{2txv}$$

$$Lu + Lv = \underline{uu'_t} + \underline{2txu} + \underline{vv'_t} + \underline{2txv}$$

Thus $Lu + Lv \neq L(u+v)$ and we conclude that the equation at hand is non-linear.

4. Classification of a PDE

A general linear equation of second order can be written

$$(1)\quad a(x,t)\,u''_{tt} + b(x,t)\,u''_{xt} + c(x,t)\,u''_{xx} + d(x,t)\,u'_t + e(x,t)\,u'_x + g(x,t)\,u = f(x,t) \qquad (x,t)\in D$$

Put

$$D(x,t) = (b(x,t))^2 - 4\,a(x,t)\,c(x,t).$$

We say that (1) is

<u>elliptic</u> if $D(x,t) < 0$ in D,
<u>parabolic</u> if $D(x,t) = 0$ in D,
<u>hyperbolic</u> if $D(x,t) > 0$ in D.

Ex 17: Consider the two dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0$$

Here is $D(x,y) = 0^2 - 4\cdot 1\cdot 1 < 0$ and the equation is elliptic.

Ex 18: Consider the heat conduction equation

$$u'_t - u''_{xx} = 0.$$

Here is $D(x,y) = 0 - 4(-1)\cdot 0 = 0$ and the equation is parabolic.

Ex 19: Consider the one-dimensional wave equation

$$u''_{tt} - u''_{xx} = 0$$

Here is $D(x,y) = 0 - 4\cdot 1\cdot(-1) = 4 > 0$ and the equation is hyperbolic

5. Superposition

Let

$$(*) \qquad Lu = 0$$

be a linear and homogeneous PDE. If $u_1, u_2, u_3, \dots$ are solutions to $(*)$, then also

$$u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

is a solution to $(*)$.

$$\Big(Lu = L(c_1 u_1 + \dots + c_n u_n) = c_1 \underbrace{Lu_1}_{=0} + \dots c_n \underbrace{Lu_n}_{=0} = 0 \Big)$$

This superposition principle holds also for infinite sums

$$u = c_1 u_1 + \dots + c_n u_n + \dots$$

if certain convergence conditions hold.

Continuous superposition principle:

Let $u_\alpha(x,t)$ satisfy that $Lu_\alpha = 0$ for all α, $a \le \alpha \le b$. Then also

$$u(x,t) = \int_a^b c(\alpha)\, u_\alpha(x,t)\, d\alpha$$

satisfies that $Lu = 0$

$$\Big(Lu = \int_a^b c(\alpha)\, Lu_\alpha(x,t)\, d\alpha = \int_a^b c(\alpha)\, 0\, d\alpha = 0 \Big)$$

EX 20: It is easy to verify that

$$u_\alpha(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{(x-\alpha)^2}{4kt}\right), \quad t > 0, -\infty < \alpha$$

satisfy the heat equation

$$(1) \qquad u'_t - k\, u''_{xx} = 0.$$

Thus

$$u(x,t) = \int_{-\infty}^{\infty} c(\alpha) \frac{1}{\sqrt{4\pi k t}} \exp - \frac{(x-\alpha)^2}{4kt}\, d\alpha$$

[illegible]

6. Well-posed Problems

A Boundary value problem is well-posed if

(a) there is a solution.
(b) the solution is unique.
(c) the solution is stable.

EX 21: Consider

(1) $$u''_{tt} + u''_{xx} = 0 \quad , t>0, -\infty<x<\infty$$

subject to the initial conditions

(2) $$u(x,0)=0 \; , u'_t(x,0)=0 \quad -\infty<x<\infty$$

The solution is $u(x,t)\equiv 0$, $t\geq 0$, $-\infty<x<\infty$

We change (2) a little to

(2') $$u(x,0)=0 \quad u'_t(x,0)=10^{-4}\sin 10^4 x$$

Then the solution is

$$u(x,t)=10^{-8}\sin(10^4 x)\sinh(10^4 t)$$

For large t $\sinh 10^4 t$ behaves like $\frac{1}{2}\exp(10^4 t)$ and we get a dramatic change in the solutions with this small perturbation of the initial data.

Thus the problem is ill-posed because (c) does not hold.

Ex 22: The boundary value problem

$$\begin{cases} u'_t - k u''_{xx} = 0, \quad 0<x<l, \ 0<t<T, \\ u(x,0) = f(x), \quad 0<x<l, \\ u(0,t) = g(t), \ u(l,t) = h(t), \quad 0<t<T, \end{cases}$$

where $f \in C[0,l]$ and $g, h \in C[0,T]$ has a unique solution $u(x,t)$ on the rectangle $R: 0 \le x \le l, 0 \le t \le T$.

Proof Later on we construct a solution of the problem!

Now assume that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem. Then, in particular the function $w(x,t) = u_1(x,t) - u_2(x,t)$ must satisfy the BVP

$$w'_t - k w''_{xx} = 0, \quad 0<x<l, \ 0<t<T \quad (1)$$

$$w(x,0) = 0, \quad 0<x<l \quad (2)$$

$$w(0,t) = w(l,t) = 0, \quad 0<t<T \quad (3)$$

We define the "energy integral"

$$E(t) = \int_0^l w^2(x,t)\,dx.$$

We note ~~that~~ $E(t) \ge 0$, $E(0) = 0$ and

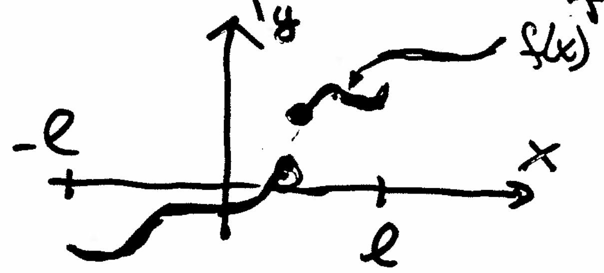
$$E'(t) = \int_0^l 2 w w'_t \,dx = 2k \int_0^l w w''_{xx} \,dx =$$

$$= \underbrace{[2k w w'_x]_0^l}_{=0} - 2k \int_0^l (w'_x)^2 \,dx \le 0$$

Thus $E(t) \le 0$ and we conclude that $E(t) = 0$ which means that $w(x,t) = u_1(x,t) - u_2(x,t) = 0$ and we get a contradiction which proves that we can have only one sol...

Some remarks on the theory of FOURIER SERIES

Consider a function $f(x)$, $-l < x < l$



The Fourier coefficients of f are defined as follows

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin n\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

The Fourier series of f is defined as

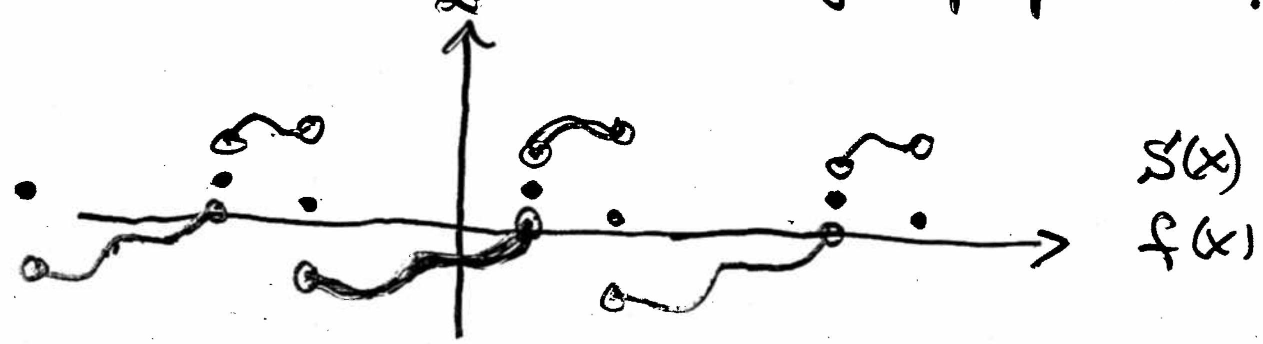
$$S(x) = a_0 + \sum_{1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

Theorem Let $f(x)$ be piecewise smooth. Then

(a) $S(x) = S(x + 2l)$ $\forall x$.

(b) $S(x) = f(x)$ in continuity points in $-l < x < l$

(c) $S(x) = \frac{f(x+) + f(x-)}{2}$ in "jump-points".



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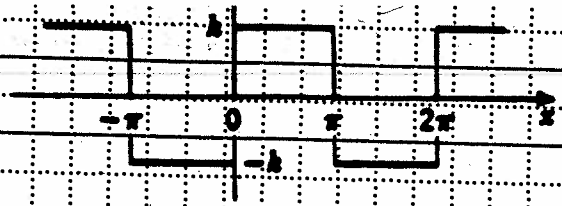
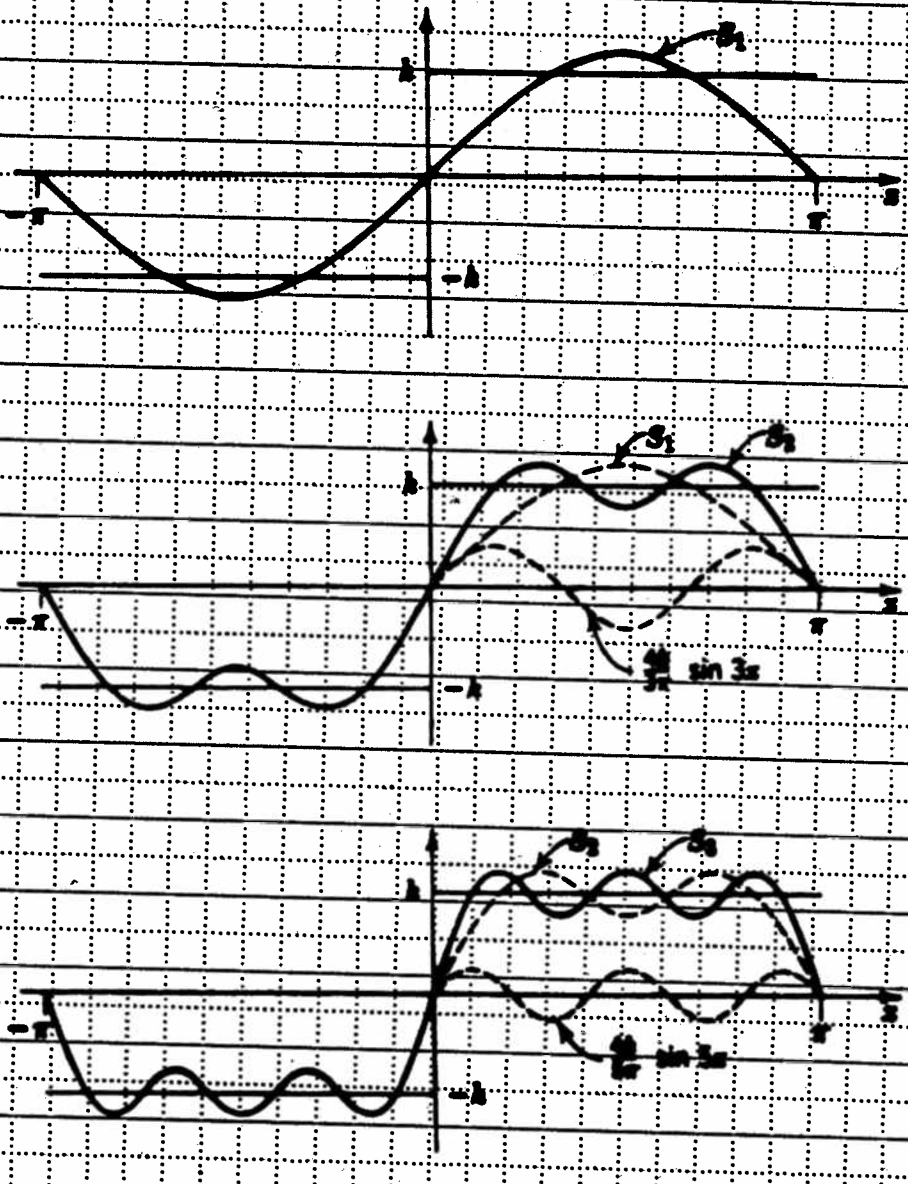
$$f(x) = \begin{cases} k, & 0 < x < \ell \\ -k, & -\ell < x \le 0 \end{cases}$$



(a) The given function $f(x)$ (Periodic square wave)



(b) The first three partial sums of the corresponding Fourier series

Fig. 239. Example 1

Now, $\cos \pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$ etc., in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots$$

and since the a_n are zero, the Fourier series of $f(x)$ is

(8) $$S(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Separation of Variables

= Fouriers method (Fourier 1768–1830) see

Modelexample: Solve

$$u'_t - k u''_{xx} = 0 \quad , 0 < x < l \, , t > 0, \quad (1$$

$$u(x,0) = f(x) \quad , 0 < x < l, \quad (2$$

$$u(0,t) = u(l,t) = 0, \quad t > 0. \quad (3$$

Summing up, the following functions solve (1) (

$$u(x,t) = \sum_1^\infty b_n \sin\frac{n\pi x}{l} \exp\left(-\frac{n^2\pi^2 k t}{l^2}\right)$$

b_n must be chosen so that

$$u(x,0) = f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l} \qquad (*)$$

Let $l = \pi$ (for simplicity)

Ex 24: $f(x) = 2\sin x + 4\sin 3x$

then $(*)$ holds if $b_1 = 2$, $b_2 = 0$, $b_3 = 4$, $b_4 = b_5 = \dots = 0$

The solution of the modelexample is

$$u(x,t) = 2\sin x \, e^{-kt} + 4\sin 3x \, e^{-9kt}$$

Ex 25: $f(x) = 1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$

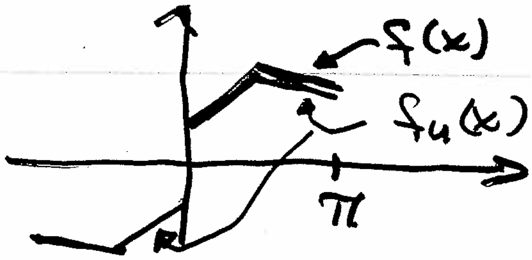
Then $(*)$ holds if $b_1 = \frac{4}{\pi}$, $b_2 = 0$, $b_3 = \frac{4}{\pi}\frac{1}{3}$, $b_4 = 0$, $b_5 = \frac{4}{\pi}\frac{1}{5}$, $b_6 = 0$, etc ...

The solution of the modelexample is

$$u(x,t) = \frac{4}{\pi}\left(\sin x \, e^{-kt} + \frac{1}{3}\sin 3x \, e^{-9kt} + \frac{1}{5}\sin 5x \, e^{-25 \dots}\right.$$

$$+ \dots) = \frac{4}{\pi} \sum^\infty \frac{1}{\dots} \dots e^{-(2n-1)^2 kt}$$

EX 26: General $f(x)$!



The solution of our model example is

$$u(x,t) = \sum_{1}^{\infty} b_n \sin nx \exp(-n^2 k t)$$

where

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_u(x) \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

which models heat flow in a homogeneous bar with constant diffusivity k whose ends are held fixed at zero degrees and whose initial temperature distribution is $f(x)$. The first step is to assume a solution of the form

STEP1 :

$$u(x,t) = X(x)T(t) \tag{4}$$

that is, a product of a function of x and a function of t. When (4) is substituted into the partial differential equation (1) the result can be written as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

with the variables x and t separated, and hence the name *separation of variables*. As x and t vary the only way a function of t can be equal to a function of x is if both functions are equal to the same constant, which we call $-\lambda$. Hence

$$X''(x) + \lambda X(x) = 0 \tag{5}$$

and

$$T'(t) = -\lambda k T(t) \tag{6}$$

Calling the constant $-\lambda$ does not mean it is negative; we use the negative sign only for convenience. Therefore the assumption (4) has led to a pair of ordinary differential equations (5) and (6). Now we apply (4) to the boundary conditions (3) and obtain

$$T(t)X(0) = 0, \qquad T(t)X(l) = 0 \tag{7}$$

Excluding the uninteresting possibility that $T(t) = 0$, we get $X(0) = 0$ and $X(l) = 0$. Therefore we are led to a boundary value problem for the function X

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < l \tag{8a}$$

$$X(0) = 0, \qquad X(l) = 0 \tag{8b}$$

In this last step, in order to get the boundary conditions on X it was essential that zero appear on the right sides in (7); hence, the boundary conditions in the original problem had to be homogeneous.

The plan can be described briefly as follows. We determine values of λ for which (8) has nontrivial solutions. Often there will be infinitely many such values $\lambda_1, \lambda_2, \ldots$ that accomplish this and to each will correspond a solution $X_n(x)$, $n = 1, 2, \ldots$, of (8) and a corresponding solution $T_n(t)$ of (6). Thus we will have constructed infinitely many solutions $u_n(x,t) = X_n(x)T_n(t)$, $n = 1, 2, \ldots$, to the partial differential equation that satisfy the boundary condi-

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tions. We then superimpose these solutions by defining

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$$

and we choose the constants b_n so that $u(x,t)$ satisfies the initial condition as well. Being a sum of solutions to a linear problem we expect u to be a solution and to satisfy the homogeneous boundary conditions.

STEP 2 To solve (8) we examine three cases: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

(i) $\lambda < 0$. If $\lambda < 0$, then the general solution of (8a) is

$$X(x) = A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x$$

where A and B are arbitrary constants. $X(0) = 0$ implies $A = 0$ and $X(l) = 0$ implies $B = 0$. Therefore the boundary value problem (8) has only the trivial solution if $\lambda < 0$.

(ii) $\lambda = 0$. In this case the general solution of (8a) is $X = Ax + B$. Application of the boundary conditions (8b) forces $A = B = 0$. Again no nontrivial solutions exist.

(iii) $\lambda > 0$. Here the general solution of (8a) is

$$X(x) = A\sin\sqrt{\lambda}\,x + B\cos\sqrt{\lambda}\,x$$

$X(0) = 0$ forces $B = 0$. Then $X(l) = 0$ implies

$$A\sin\sqrt{\lambda}\,l = 0$$

Now we have the possibility of selecting values of λ that will make this equation hold without choosing $A = 0$, which would again lead to a trivial solution. Therefore take

$$\sqrt{\lambda}\,l = n\pi, \qquad n = 1,2,3,\ldots$$

or

$$\lambda = \frac{n^2\pi^2}{l^2}, \qquad n = 1,2,3,\ldots \tag{9}$$

The case $n = 0$ was the subject of Case (ii) above. Corresponding to each value of λ in (9) there is a solution of (8) given by

$$X_n(x) = \sin\frac{n\pi x}{l}, \qquad n = 1,2,\ldots \tag{10}$$

We have chosen the constant A in front of the sine function as 1 ...

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad n = 1, 2, \ldots \tag{11}$$

for which (8) has a nontrivial solution are called the *eigenvalues* and the corresponding solutions (10) are called the *eigenfunctions*. Each choice of λ_n provides a solution to the T equation (6). Equation (6) becomes

$$T_n'(t) = -\frac{n^2\pi^2 k}{l^2} T_n(t), \qquad n = 1, 2, \ldots$$

from which we infer

$$T_n(t) = \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right), \qquad n = 1, 2, \ldots$$

Step 3

As indicated we attempt to find a solution to the original boundary value problem (1)–(3) of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \tag{12}$$

The constants b_n are determined from the initial condition (2), which becomes

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{13}$$

We proceed formally. Let m be a fixed but arbitrary positive integer. Multiplying (13) by $\sin(m\pi x/l)$ and integrating the result from 0 to l gives

$$\int_0^l f(x) \sin\frac{m\pi x}{l}\,dx = \sum_{n=1}^{\infty} b_n \int_0^l \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\,dx \tag{14}$$

In obtaining (14) we have interchanged the order of integration and summation. From calculus, for positive integers m and n

$$\int_0^l \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\,dx = \begin{cases} 0, & n \neq m \\ \dfrac{l}{2}, & n = m \end{cases} \tag{15}$$

STEP 4 b_n are calculated by using some identification e.g. the theory of Fourier series.

Problems - Lecture 4

1. By introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, find the general solution of the equation

$$y u'_x - x u'_y = 0.$$

Ans: $u = f(x^2+y^2)$, where f is an arbitrary function.

2. a) Determine the regions where the following equations are hyperbolic, elliptic or parabolic:

a) $t u''_{tt} + u''_{xx} = 0$ b) $u''_{tt} + (1+x^2) u'_x - u'_t = \cos$

c) $u''_{tt} - u''_{xx} + u = e^t$ d) $u''_{tx} + u^2 = \sin t.$

b) Determine whether the following equations are linear or nonlinear

e) $u'_t u''_{tt} + 3ut = 1$ f) $u''_{tt} - u''_{xx} = t + e^t$

g) $e^t u''_{tx} - x^2 u = \sin t$ h) $u''_{tx} + u^2 = \arctan tx.$

3.* Using the energy-method prove that the solutions to the bounary value problem

$$\begin{cases} u'_t - k u''_{xx} = 0, \; 0 < x < \ell, \; 0 < t < T, \\ u(x,0) = f(x), \; 0 < x < \ell, \\ u'_x(0,t) = 0, \; u'_x(\ell,t) = g(t), \; 0 < t < T, \end{cases}$$

are unique for any $T > 0$.

4. A radially symmetric solution to the Laplace equation $\Delta u = 0$ in R^3 is a solution of the form

$$u = u(r), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

where u just depends on the distance r from origin. Find all radially symmetric solutions of $\Delta u = 0$ in R^3. Solve also the corresponding problem in R^2.

5. a) Find the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$. To which function on the real line R does the Fourier series converge?

b) Use a) to prove that π can be written:

$$\pi = 12\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right)$$

6. Use the Fourier method to solve the following problem:

$$\begin{cases} u''_{tt} = c^2 u''_{xx} - a^2 u, & 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0, & t > 0, \\ u(x,0) = f(x),\ u'_t(x,0) = 0, & 0 < x < l. \end{cases}$$

7.* Use the Fourier method to solve the following problem:

$$\begin{cases} u'_t - u''_{xx} = 0, & 0 < x < l,\ t > 0, \\ u'_x(0,t) = u'_x(l,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < l. \end{cases}$$