Lecture 5

First we make some preparations of independent interest

1. Cauchy's equation

(*) $\quad x^2y'' + axy' + by = 0$

Sol: Put $y(x) = x^r$. Then

$$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}.$$

Insert into (*) and we get

$$r(r-1)x^r + arx^r + bx^r = 0$$

$\Leftrightarrow$

(**) $\quad \boxed{r(r-1) + ar + b = 0}$

Three cases:

1°. (**) has two real roots r_1 and r_2, $r_1 \neq r_2$. Then

$$y(x) = Ax^{r_1} + Bx^{r_2}.$$

2°. (**) has a double root $r_1 = r_2 = r$. Then

$$y(x) = Ax^r + B\ln x\, x^r.$$

3°. (**) has the roots $\alpha \pm i\beta$. Then (*) has the complex solutions $\alpha \pm i\beta$

(***) $\quad y(x) = Ax^{\alpha+i\beta} + Bx^{\alpha-i\beta}.$

Remark:

$$x^{\alpha+i\beta} = x^\alpha x^{i\beta} = x^\alpha e^{i\beta\ln x} = x^\alpha(\cos(\beta\ln x) + i\sin(\beta\ln x))$$

$$x^{\alpha-i\beta} = \cdots\cdots = x^\alpha(\cos(\beta\ln x) - i\sin(\beta\ln x))$$

Therefore (***) can be written

$$y(x) = x^{\alpha}\left((A+B)\cos(\beta \ln x) + i(A-B)\sin(\beta \ln x)\right)$$

consider now only constants A and B so that

$$C = A+B \quad \text{and} \quad D = i(A-B)$$

are <u>real</u> numbers. Then

$$y(x) = x^{\alpha}\left(C\cos(\beta \ln x) + D\sin(\beta \ln x)\right)$$

is a <u>real solution</u> of (*).

<u>Example 1</u>: Solve

$$x^2 y'' + 2xy' - 6y = 0.$$

<u>Sol</u>: The characteristic equation is

$$r(r-1) + 2r - 6 = 0, \text{ i.e.,}$$

$$r^2 + r - 6 = 0$$

$$r_1 = 2, \quad r_2 = -3$$

$$\therefore \quad \underline{y(x)} = Ax^2 + Bx^{-3}$$

<u>Example 2</u>: Solve

$$x^2 y'' + 2xy' + \lambda y = 0, \quad \lambda > \frac{1}{4}$$

<u>Sol</u>: The characteristic equation is

$$r(r-1) + 2r + \lambda = 0, \text{ i.e.,}$$

$$r^2 + r + \lambda = 0$$

$$r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}}$$

$$\therefore \underline{y(x)} = \frac{A}{\sqrt{x}}\sin\left(\sqrt{\lambda - \frac{1}{4}}\,\ln x\right) + \frac{B}{\sqrt{x}}\cos\left(\sqrt{\lambda - \frac{1}{4}}\,\ln x\right)$$

2. Introductory examples of "Sturm-Liouville problems"

Example 3: Solve

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(\ell) = 0 . \end{cases}$$

Sol: We found before that this problem can be solved iff

$$\lambda = \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \; n = 1, 2, 3, \dots \quad (\underline{\text{eigen values}})$$

with the corresponding solutions

$$y_n = a_n \sin \frac{n\pi}{\ell} x \quad (\underline{\text{eigenfunctions}})$$

Example 4: Solve

$$\begin{cases} X''(x) - \lambda X(x) = 0 & 0 \le x \le 1 \\ X(0) = 0 \\ X'(1) = -3X(1) . \end{cases}$$

Sol: Three cases

$\underline{\lambda = 0}$: $X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$

$X'(1) = -3X(1) \Rightarrow A = -3A \Rightarrow A = 0$

$\therefore X(x) \equiv 0$ not interesting!

$\underline{\lambda = p^2 > 0}$: $X(x) = Ae^{px} + Be^{-px}$

$X(0) = 0 \Rightarrow A = -B$

$X'(1) = -3X(1) \Rightarrow p(Ae^{p} - Be^{-p}) =$

$$= -3(Ae^{p} + Be^{-p}) \Leftrightarrow \begin{cases} (p+3)A = 0 \\ (p+3)B = 0 \end{cases} \Rightarrow A = B = 0 \quad \boxed{p \neq -3}$$

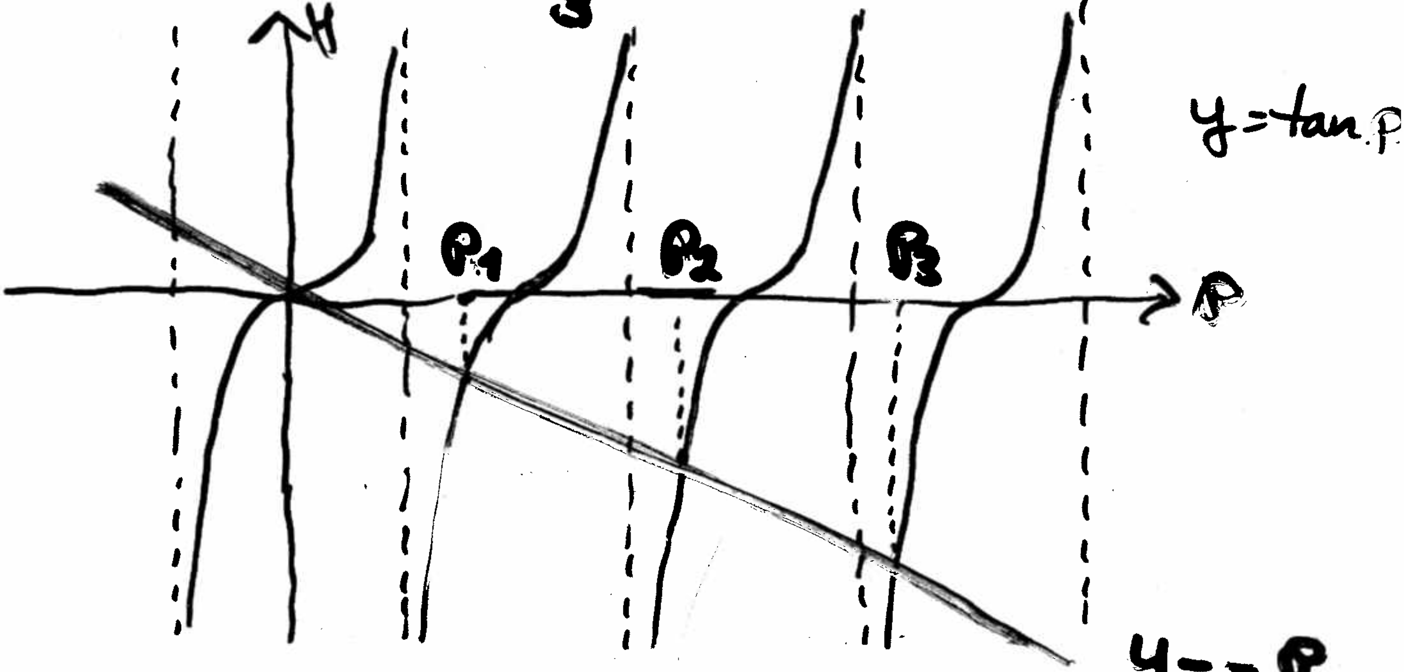
$\therefore X(x) \equiv 0$ not interesting!

$\lambda = -p^2 < 0$: $X(x) = A\cos px + B\sin px$

$$X(0) = 0 \Rightarrow A = 0$$

$$X'(1) = -3X(1) \Rightarrow Bp\cos p = -3B\sin p$$

$$\Leftrightarrow \tan p = -\frac{p}{3}$$



We find that we have solutions iff

$$p = p_n, \quad n = 1, 2, 3, \ldots \quad \text{(eigenvalues)}$$

(where p_n are the solution of $\tan p = -\frac{p}{3}$)

with the corresponding solutions

$$X_n(x) = a_n \sin p_n x \quad \text{(eigenfunctions)}$$

Example 6: Solve

$$\begin{cases} x^2 X''(x) + 2xX'(x) + \lambda X = 0, \\ X(1) = 0, X(e) = 0. \end{cases}$$

Sol: We must consider the cases $\lambda < 1/4$, $\lambda = 1/4$ and $\lambda > \frac{1}{4}$ (see Example 2)

The cases $\lambda < 1/4$ and $\lambda = \frac{1}{4}$ only give the trivial solution $X(x) \equiv 0$ (Prove that!)

For the case $\lambda > \frac{1}{4}$ we have

$$X(x) = \frac{A}{\sqrt{x}} \sin\left(\sqrt{\lambda - \tfrac{1}{4}}\,\ln x\right) + \frac{B}{\sqrt{x}} \cos\left(\sqrt{\lambda-\tfrac{1}{4}}\,\ln x\right)$$

$X(1) = 0 \Rightarrow B = 0$

$X(e) = 0 \Rightarrow \sin\sqrt{\lambda - \frac{1}{4}} = 0$ (or $A = 0$)

$\Rightarrow \sqrt{\lambda - \frac{1}{4}} = n\pi$, $n = 1, 2, 3, \ldots \Rightarrow$

$\lambda = (n\pi)^2 + \frac{1}{4}$, $n = 1, 2, 3, \ldots$.

we find that we have solutions iff

$\lambda = \lambda_n = (n\pi)^2 + \frac{1}{4}$, $n = 1, 2, 3, \ldots$ (eigenvalues)

with the corresponding solutions

$X = X_n(x) = \frac{a_n}{\sqrt{x}} \sin(n\pi \ln x)$ (eigenfunctions)

<u>Example 7</u>: (Bessel's equation, see Appendix 1)

$$\begin{cases} \dfrac{d^2 w}{dr^2} + \dfrac{1}{r}\dfrac{dw}{dr} + k^2 w = 0 \\ w(R) = 0, \quad w'(r) \text{ is finite.} \end{cases}$$

Sol: A general solution (see Appendix 1) is

$$w = C_1 J_0(kr) + C_2 Y_0(kr),$$

where J_0 and Y_0 are the Bessel functions of the first and the second kind, respectively.

$W'(r)$ is finite $\Rightarrow C_2 = 0$

$$W(R) = C_1 J_0(kR) = 0 \quad ?$$

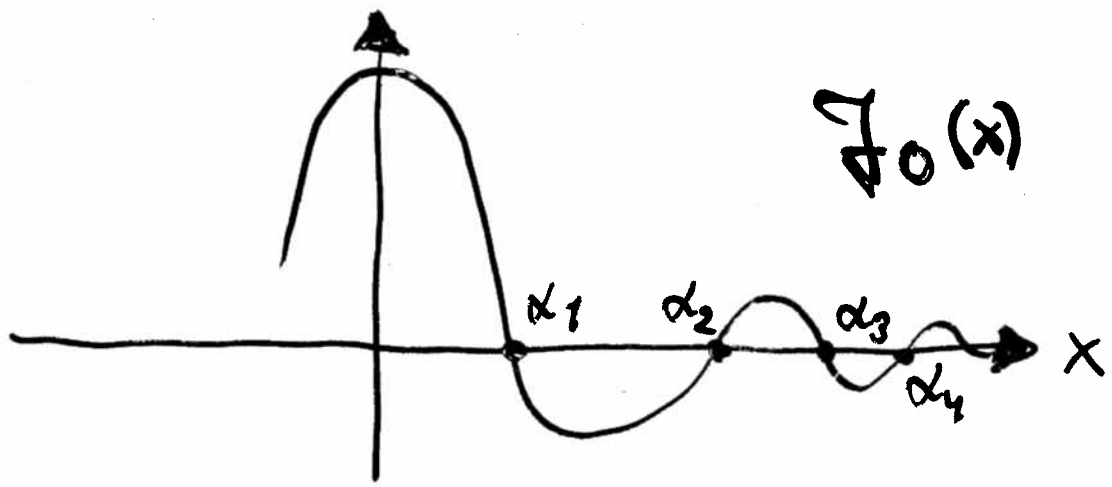
It is well-known (see Appendix) that J_0 has infinite many real zeros α_n ($\alpha_1 = 2.4048$, $\alpha_2 = 5.5201$, $\alpha_3 = 8.6537$, $\alpha_4 = 11.7915$, etc....)

we find that we have solutions iff

$$k_n = \frac{\alpha_n}{R} \; , \; n = 1, 2, 3, \ldots \quad \text{(eigenvalues)}$$

with the corresponding solutions

$$W_n(r) = J_0\left(\frac{\alpha_n}{R} r\right) \quad \text{(eigenfunctions)}$$



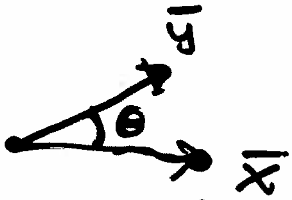
3. Inner product and norm

Case 1: (vectors)

$\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$

$\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2$ (inner product)

$|\bar{x}|^2 = x_1^2 + x_2^2$ (norm)



$\bar{x} \cdot \bar{y} = |\bar{x}|\,|\bar{y}| \cos\theta$

- $|\bar{x} - \bar{y}|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$ (distance)

- $\bar{x} \perp \bar{y} \Leftrightarrow \bar{x} \cdot \bar{y} = 0$ (orthogonality)

Case 2: functions on $[0, \ell]$: $f(x), g(x)$
($r(x)$ is a weight function, $r(x) > 0$)

$\langle f, g \rangle = \int_0^\ell f(x) g(x) r(x)\, dx$ (inner product)

$\|f\|^2 = \int_0^\ell |f(x)|^2 r(x)\, dx$ (norm)

- $\|f - g\| = \int_0^\ell |f(x) - g(x)|^2 r(x)\, dx$ (distance)

$f \perp g \Leftrightarrow \langle f, g \rangle = 0 \Leftrightarrow \int_0^\ell f(x) g(x) r(x)\, dx = 0$
(orthogonality)

4. Sturm-Liouville Problems

$$(p(x)y')' + (-q(x) + \lambda r(x))\, y = 0, \quad 0<x<\ell,$$

$$c_1 y(0) + c_2 y'(0) = 0$$

$$c_3 y(\ell) + c_4 y'(\ell) = 0$$

Example 8: $r(x)=1$, $p(x)=1$, $q(x)=0$, $c_2=0$, $c_4=0$

$$\begin{cases} y''+\lambda y = 0, \\ y(0)=0, \\ y(\ell)=0. \end{cases}$$

$\boxed{\text{compare with Example 3}}$

In this case we have

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n=1,2,3,\ldots$ (eigen values)

$y_n = \sin\frac{n\pi}{\ell}x$ (eigen functions)

$$\langle y_n, y_m\rangle = \int_0^\ell \sin\frac{n\pi}{\ell}x \sin\frac{m\pi}{\ell}x\,dx = 0 \quad n\neq m$$

$$\|y_n\|^2 = \int_0^\ell \sin^2\frac{n\pi}{\ell}x\,dx = \int_0^\ell \frac{1-\cos\frac{2n\pi}{\ell}x}{2}\,dx = \frac{\ell}{2}$$

$$(*)\quad C_n = \frac{1}{\|y_n\|^2}\langle f, y_n\rangle = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi}{\ell}x\,dx$$

$$S(x) = \sum_1^\infty C_n \sin\frac{n\pi}{\ell}x,$$

where C_n are the Fourier coefficients in $(*)$.

Example 9: Examples 4-7 are also Sturm-Liouville problems.

Theorem [redacted] *For a regular Sturm-Liouville problem*

(i) *The eigenvalues are real and to each eigenvalue there corresponds a single eigenfunction unique up to a constant multiple.*

(ii) *The eigenvalues form an infinite sequence $\lambda_1, \lambda_2, \ldots$ and can be ordered according to*

$$0 \le \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with

$$\lim_{n \to \infty} \lambda_n = \infty$$

(iii) *If $y_1(x)$ and $y_2(x)$ are two eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2, then*

$$\langle y_1, y_2 \rangle = \int_0^l y_1(x) y_2(x) r(x)\, dx = 0$$

5. Generalized Fourier expansion

- Let $f(x) = \sum_{1}^{\infty} c_n y_n(x)$, where $y_n \perp y_m$, $n \neq m$. Then

$$\langle f, y_m \rangle = \langle \sum_{n=1}^{\infty} c_n y_n, y_m \rangle = \sum_{n=1}^{\infty} c_n \langle y_n, y_m \rangle =$$
$$= c_m \|y_m\|^2$$

- Let f be <u>any</u> function on $[0, \ell]$. Then we define <u>the generalized Fourier series of f</u> as

$$S(x) = \sum_{1}^{\infty} c_n y_n(x),$$

where

$$c_n = \frac{1}{\|y_n\|^2} \cdot \langle f, y_n \rangle$$

are <u>the generalized Fourier coefficients</u>.

<u>THEOREM</u>: Let $y_1, y_2, \ldots$ be an orthonormal set of eigenfunctions for the regular S-L.-problem and let f be piecewise smooth. Then, for every x in $[0, \ell]$,

(a) $S(x) = f(x)$, if f is continuous in x

(b) $S(x) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$ if f has a jump in x

6. Some applications

Example 10 A rod extending from $x=0$ to $x=\ell$ has end temperatures given by

$$u(0,t) = u(\ell,t) = 0, \; t > 0,$$

and an initial temperature distribution given by

$$u(x,0) = f(x), \; 0 \le x \le \ell.$$

Find the temperature $u(x,t)$ at point x, $0 \le x \le \ell$, in time t, $t \ge 0$.

Remark: The rod has constant density ρ, specific heat c_v and also constant thermel conductivity K.

Solution: We have seen (see page 16, lecture 1) that the mathematical modelling of this problem is as follows:

$$(*) \quad \begin{cases} u'_t(x,t) - k\, u''_{xx}(x,t) = 0, \; 0 \le x \le \ell, \; t \ge 0 \\ u(0,t) = u(\ell,t) = 0, \; t > 0, \\ u(x,0) = f(x), \; 0 \le x \le \ell. \end{cases} \qquad \boxed{k = \frac{K}{c_v \rho}}$$

First we do the following natural scaling of $(*)$ (see page 15, lecture 1):

$$\bar{t} = \frac{k}{\ell^2} t, \quad \bar{x} = \frac{x}{\ell},$$

Then we have to solve the following standard problem

$$\begin{cases} (1) & \tilde{u}'_{\bar{t}}(\bar{x},\bar{t}) - \tilde{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = 0, \quad 0 \le \bar{x} \le 1, \bar{t} \ge 0 \\ (2) & \tilde{u}(0,\bar{t}) = u(1,\bar{t}) = 0, \quad \bar{t} \ge 0 \\ (3) & \tilde{u}(\bar{x},0) = \tilde{f}(\bar{x}), \quad 0 \le \bar{x} \le 1 \end{cases}$$

$$\boxed{\tilde{f}(\bar{x}) = f(\bar{x} \cdot \ell)}$$

Fourier's method:

Step 1: We try to find solutions of (1) of the type

$$\tilde{u}(\bar{x},\bar{t}) = X(\bar{x})\, T(\bar{t}).$$

Inserting this into (1) gives

$$\frac{T'(\bar{t})}{T(t)} = \frac{X''(\bar{x})}{X(\bar{x})} = -\lambda$$

i.e that

$$X''(\bar{x}) + \lambda X(\bar{x}) = 0 \qquad (1)$$

and

$$T'(\bar{t}) + \lambda T(\bar{t}) = 0 \qquad (2)$$

we also try to get (2) satisfied i.e.

$$X(0)T(\bar{t}) = X(1)\, T(\bar{t}) = 0 .$$

Excluding the uninteresting trivial case $T(\bar{t}) \equiv 0$ we conclude that $X(0) = X(1) = 0$ and this together with (1) leads to the Sturm-Liouville

Problem (se Example 8)

$$(**)\quad \begin{cases} X''(\bar{x}) + \lambda X(\bar{x}) = 0 \\ X(0) = X(1) = 0 \end{cases}$$

Step 2: Three cases: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

$\lambda < 0$ Gives only the solution $X(\bar{x}) = 0$

$\lambda = 0$ Gives only the solution $X(\bar{x}) = 0$

$\lambda > 0$. We have

$$X(\bar{x}) = A \sin\sqrt{\lambda}\,\bar{x} + B\cos\sqrt{\lambda}\,\bar{x}$$

$$X(0) = 0 \Rightarrow B = 0$$
$$X(1) = 0 \Rightarrow A = 0 \ \underline{\text{or}}\ \sqrt{\lambda} = n\pi$$

Hence the S.L.-problem $(**)$ has the eigenvalues: $\lambda_n = (n\pi)^2$, $n = 1, 2, \ldots$ and (corresponding) eigenfunctions $X_n(\bar{x}) = \sin n\pi\bar{x}$

Moreover, for these values $\lambda = \lambda_n$ the equation (2) has the solutions:

$$T(\bar{t}) = T_n(\bar{t}) = e^{-(n\pi)^2 \bar{t}}$$

We conclude that

$$\tilde{u}_n(\bar{x}, \bar{t}) = \sin n\pi\bar{x} \cdot e^{-(n\pi)^2 \bar{t}}$$

are all solutions satisfying 1 and 2.

Step 3: The superposition principle thus implies that also

$$■ \quad \tilde{u}(\bar{x},\bar{t}) = \sum_{n=1}^{\infty} \tilde{b}_n \sin n\pi\bar{x}\, e^{-(n\pi)^2 \bar{t}}$$

satisfies 1 and 2. We now try to get also 3 satisfied with this function ■ and obtain that

$$\tilde{u}(\bar{x},0) = \sum_{n=1}^{\infty} \tilde{b}_n \sin n\pi\bar{x}$$

Moreover, by choosing $\tilde{b}_n$ as the Fourier coefficients, i.e.

$$✦ \qquad \tilde{b}_n = 2\int_0^1 \tilde{f}(\bar{x}) \sin n\pi\bar{x}\, d\bar{x},$$

in fact this equality holds and we conclude that

$\tilde{u}(\bar{x},\bar{t})$ defined by ■ with b_n as in ✦ in fact satisfies 1, 2, 3.

<u>Final step</u>: By now using our scaling conditions ● we obtain also the solution of our original problem described by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}\, e^{-\frac{(n\pi)^2}{\ell^2} k t}$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin \frac{n\pi}{\ell} x\, dx.$$

Example 11: [redacted] A rod extending from $x = 1$ to $x = e$ has end temperature given by

$$u(1,t) = u(e,t) = 0, \qquad t > 0$$

and an initial temperature distribution given by

$$u(x,0) = f(x), \qquad 1 < x < e$$

No sources are present and the rod has constant density ρ and specific heat c_v but its thermal conductivity K varies according to $K(x) = x^2$. [redacted] The equation governing the temperature $u(x,t)$ is

$$c_v \rho u_t = \frac{\partial}{\partial x}(x^2 u_x), \qquad 1 < x < e, \quad t > 0 \qquad (1)$$ [redacted]

To apply the Fourier method to find the solution let $u = X(x)T(t)$. Substituting into (32) and separating variables gives

$$c_v \rho \frac{T'}{T} = \frac{1}{X}\frac{d}{dx}(x^2 X') = -\lambda$$

where $-\lambda$ is constant and

$$X(1) = X(e) = 0 \qquad (2)$$ [redacted]

Thus T satisfies the equation

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$$T' = -\frac{\lambda}{c_v \rho} T$$

(3) [redacted]

and X satisfies

$$\frac{d}{dx}(x^2 X') + \lambda X = 0, \qquad 1 < x < e$$

(4) [redacted]

The ordinary differential equation (4) [redacted] and boundary conditions (2) [redacted] define a regular Sturm-Liouville problem on $[1, e]$. To determine the eigenvalues and eigenfunctions we rewrite [redacted] (4) as

$$x^2 X'' + 2xX' + \lambda X = 0$$

See Example6

and recognize it as a Cauchy-Euler equation with auxiliary equation $m(m-1) + 2m + \lambda = 0$. The roots are

$$m = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \lambda}$$

(5) [redacted]

If $\lambda = \frac{1}{4}$, then the roots are $m = -\frac{1}{2}, -\frac{1}{2}$, and 4 [redacted] has general solution $X = (A + B \ln x)x^{-1/2}$. Applying the boundary conditions (33) gives $A = B = 0$, and therefore there are only trivial solutions. If $\lambda < \frac{1}{4}$, then the roots [redacted] (5) are real and the general solution of [redacted] (4) is

$$X = Ax^{-\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}} + Bx^{-\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}}$$

Applying the boundary conditions again gives $A = B = 0$. In the case $\lambda > \frac{1}{4}$ the roots are complex and [redacted] (4) has general solution

$$X = \frac{A}{\sqrt{x}} \sin\left(\sqrt{\lambda - \tfrac{1}{4}}\, \ln x\right) + \frac{B}{\sqrt{x}} \cos\left(\sqrt{\lambda - \tfrac{1}{4}}\, \ln x\right).$$

The boundary condition $X(1) = 0$ forces $B = 0$. Then $X(e) = 0$ becomes

$$\frac{A}{\sqrt{e}} \sin\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = 0$$

which forces

$$\sqrt{\lambda - \tfrac{1}{4}} = n\pi, \qquad n = 1, 2, \ldots$$

$n = 0$ is the case $\lambda = \frac{1}{4}$, which was discussed above). Therefore the eigenvalues of the Sturm-Liouville problem (35) and (33)

$$\lambda_n = n^2\pi^2 + \tfrac{1}{4}, \qquad n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$X_n(x) = \frac{1}{\sqrt{x}} \sin(n\pi \ln x), \quad n = 1, 2, \ldots$$

Example 6

For each n the equation (3) becomes

$$T_n' = -\frac{\lambda_n}{c_v \rho} T_n$$

with solution

$$T_n(t) = \exp\left(-\frac{\lambda_n}{c_v \rho} t\right), \quad n = 1, 2, \ldots$$

We conclude that

$$T_n(t)X_n(x) = \exp\left(\frac{-\lambda_n t}{c_v \varrho}\right)\frac{1}{\sqrt{x}}\sin(n\pi \ln x)$$

are solutions satisfying the boundary conditions. By using superposition we find that also

$$u(x,t) = \sum_1^\infty a_n \exp\left(\frac{-\lambda_n t}{c_v \varrho}\right)\frac{1}{\sqrt{x}}\sin(n\pi \ln x)$$

is a solution satisfying the boundary conditions. Moreover

$$u(x,0) = \sum_1^\infty a_n \frac{1}{\sqrt{x}}\sin(n\pi \ln x) = f(x)$$

if

$$a_n = \frac{1}{\|X_n\|^2}\int_1^e f(x)\frac{1}{\sqrt{x}}\sin(n\pi \ln x)\,dx .$$

$$\left(\|X_n\|^2 = \left(\int_1^e \frac{1}{x}\sin^2(n\pi \ln x)\,dx\right) = \frac{1}{2}.\right)$$

We conclude that

$$u(x,t) = \sum_1^\infty a_n \exp\left(\frac{-\lambda_n t}{c_v \varrho}\right)\frac{1}{\sqrt{x}}\sin(n\pi \ln x),$$

where

$$a_n = 2\int_1^e \frac{f(x)}{\sqrt{x}}\sin(n\pi \ln x)\,dx, \quad n=1,2,\ldots$$

Example 11: Solve

$$(1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$(2) \quad u(0,t) = 0,$$

$$(3) \quad u'_x(1,t) = -3u(1,t),$$

$$(4) \quad u(x,0) = f(x).$$

Step 1: We put $u(x,t) = X(x)T(t)$ into (1) and find

$$X(x)T'(t) = X''(x)T(t)$$

$$\Leftrightarrow$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

$$\Leftrightarrow$$

$$(5) \quad T'(t) - \lambda T(t) = 0$$

$$(6) \quad X''(x) - \lambda X(x) = 0$$

Step 2: We consider three cases

$\lambda = 0$: Then, according to (5)-(6),

$$u(x,t) = T(t)X(x) = Ax + B$$

(2) $\Rightarrow$ $B = 0$. (3) $\Rightarrow$ $A = 0$ (c.f. Example 4)

Thus $u(x,t) \equiv 0$ not interesting!

$\lambda = p^2 > 0$: In a similar way we find that

$u(x,t) \equiv 0$, not interesting

$\lambda = -p^2 < 0$: In this case we find that we have the solutions

$$(*) \qquad u_n(x,t) = B_n e^{-p_n^2 t} \sin p_n x$$

where p_n are the solutions of

$$\tan p = -\frac{p}{3}$$

$(p_1 \le p_2 \le p_3 \le \ldots)$

Step 3: All functions defined by $(*)$ satisfy (1), (2) and (3). Moreover, by the superposition principle, also

$$u(x,t) = \sum_1^\infty B_n e^{p_n^2 t} \sin p_m x$$

satisfies (1), (2) and (3). Moreover,

$$u(x,0) = \sum_{n=1}^\infty B_n \sin p_n x = f(x)$$

if we choose

$$(**) \qquad B_n = \frac{(f(x), \sin(p_n x))}{\| \sin(p_n x) \|_1^2} = \frac{\int_0^1 f(x) \sin p_n x \, dx}{\int_0^1 \sin^2 p_n x \, dx}.$$

Motivation: This is a regular Sturm-Liouville problem and we can use the theory for generalized Fourier series.

Answer: $u(x,t) = \sum_1^\infty B_n e^{-p_n^2 t} \sin p_n x$,

where p_n are the positive solutions of the equation $\tan p = -\frac{p}{3}$ $(p_1 < p_2 < p_3 < \ldots)$ and B_n are defined by $(**)$.

Example 12: The wave equation; vibration of a circular membrane of radius R.

(1) $u''_{tt} = c^2(u''_{xx} + u''_{yy})$

(2) $u(R,t) = 0$ (fixed boundary)

(3) $u(r,\theta) = f(r)$ (initial deflection)

(4) $\frac{\partial u}{\partial t}(r,0) = g(r)$ (initial velocity)

Using polar coordinates, defined by $x = r\cos\theta$, $y = r\sin\theta$, we find that (1) can be written

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

or, if $u(r,t)$ is radially symmetric)

(1') $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right).$$

We put $u(r,t) = W(r)G(t)$, insert $u(r,t)$ into (1) and find that

(5) $W'' + \frac{1}{r}W' + k^2 W = 0, \quad 0 \le r \le R,$

(6) $\ddot{G} + (ck)^2 G = 0, \quad t > 0.$

Moreover, by (2),

(7) $W(R) = 0$

(5) together with (7) is a S-L problem which produces the eigenfunctions

$$W_n(r) = J_0\left(\frac{\alpha_n}{R} r\right),$$

where $\alpha_n = k_n R$ are the solutions of the equation $J_0(kR) = 0$ (see Example 7)
(Scalar product $(f,g) = \int_0^R f(r)\, g(r)\, r\, dr$)

By solving also ⑥ for these values and using superposition we find that also

(1) $$u(r,t) = \sum_1^\infty \left(A_n \cos\frac{c\alpha_n}{R} t + B_n \sin\frac{c\alpha_n}{R} t\right) J_0\left(\frac{\alpha_n}{R} r\right)$$

is a solution satisfying ① and ②.

Moreover ③ is satisfied, i.e.

$$u(r,0) = \sum_1^\infty A_n J_0\left(\frac{\alpha_n}{R} r\right) = f(r)$$

if

(2) $$A_n = \frac{1}{\int_0^R J_0^2\left(\frac{\alpha_n}{R} r\right) r\, dr} \cdot \int_0^R f(r)\, J_0\left(\frac{\alpha_n}{R} r\right) r\, dr$$

In the same way we find that ④ is satisfied if

(3) $$B_n \frac{c\alpha_n}{R} = \frac{1}{\int_0^R J_0\left(\frac{\alpha_n}{R} r\right) r\, dr} \int_0^R f(r)\, J_0\left(\frac{\alpha_n}{R} r\right) r\, dr$$

Answer The solution is (1) where A_n and B_n are defined by (2) and (3).

* More details can be found in Appendix 1.

Problems - Lecture 5

1. Find the eigenvalues and normalized eigenfunctions to the following problems

a) $-\phi'' = \lambda\phi,\ 0<x<l,\ \phi'(0) = \phi'(l) = 0,$

b) $-\phi'' = \lambda\phi,\ 0<x<l,\ \phi'(0) = \phi(l) = 0,$

c) $-(x^2\phi')' = \lambda\phi,\ 1<x<e,\ \phi(1) = \phi(e) = 0.$

2.* A rod extending from $x=1$ to $x=e$ has its ends maintained at a constant zero degrees, and the initial temperature distribution in the rod is given by $f(x)$, $1<x<e$. The rod has constant density ρ and constant specific heat C, but its thermal conductivity varies via $K = x^2$, $1<x<e$. Formulate an initial boundary value problem for the temperature $u(x,t)$ in the rod, and solve the problem by the Fourier method.

3.* Use the Fourier method to solve the following problem

$$\begin{cases} u''_{tt} = c^2 u''_{xx} - a^2 u, & 0<x<l,\ t>0, \\ u(0,t) = u(l,t) = 0, & t>0, \\ u(x,0) = f(x),\ u'_t(x,0) = 0, & 0<x<l, \end{cases}$$

4.* a) Solve the following problem:

$$\begin{cases} u'_t = 9u''_{xx} &, 0 \le x \le 1, t > 0, \\ u(0,t) = 0 &, t > 0, \\ u'_x(1,t) = -2u(1,t) &, t > 0, \\ u(x,0) = f(x) &, 0 \le x \le 1. \end{cases}$$

b) Give physical intepretation of all terms of this problem.