Lecture 6 - Transform theory

1. Transforms of Fourier serie type

EX1 (Classical form)

$$F_d : f(t) \longrightarrow \{a_0, a_1, b_1, \ldots, a_n, b_n, \ldots\}$$

where

$$a_0 = \int_0^1 f(t)\,dt$$

$$a_n = 2\int_0^1 f(t)\cos 2n\pi t\,dt,\ n=1,2,\ldots$$

$$b_n = 2\int_0^1 f(t)\sin 2n\pi t\,dt,\ n=1,2,\ldots$$

"The signal" $f(t)$ can be reconstructed (in continuity points) as follows:

$$F_d^{-1} : f(t) = a_0 + \sum_1^\infty a_n \cos 2n\pi t + b_n \sin 2n\pi t.$$

EX2: (Generalized form e.g. with eigen-functions from S.L.-problems)

Let $\{y_n(x)\}_1^\infty$ be an orthogonal system i.e.

$$\langle y_n, y_m\rangle = \begin{cases} 0 & , n \neq m \\ \|y_n\|^2 & , n = m \end{cases}.$$

Then we define

$$F_d : f(t) \longrightarrow \{a_n^*\}_1^\infty,$$

where

$$a_n^* = \frac{1}{\|y_n\|^2} \langle f, y_n \rangle .$$

(a_n^* are the Fourier coefficients)

The with fairly general assumptions on $\{y_n(x)\}_1^\infty$ the signal $f(t)$ can again be reconstructed (in continuity points) as

$$F_d^{-1} : f(t) = \sum_1^\infty c_n^* y_n(t) .$$

- Again $y_n(t)$ can be interpreted as basis functions and c_n^* are the corresponding amplitudes.

<u>Remark</u>: The classical form in Ex 1 is obtained by simply considering

$$\{y_n(t)\}_1^\infty = \{1, \cos 2\pi t, \sin 2\pi t, \ldots, \cos 2n\pi t, \sin 2n\pi t, \ldots\}$$

Note e.g that

$$\langle f, \cos 2n\pi t \rangle = \int_0^1 f(t) \cos 2n\pi t \, dt$$

and

$$\|\cos 2n\pi t\|^2 = \int_0^1 \cos^2 2n\pi t \, dt =$$

$$\int_0^1 \frac{1+\cos 4n\pi t}{2} dt = \frac{1}{2} .$$

Ex3: (Complex form)

$$F_c : f(t) \longrightarrow \{c_n\}_{-\infty}^{\infty},$$

where

$$c_0 = \int_0^1 f(t)\,dt,$$

$$c_n = \int_0^1 f(t)\, e^{-i2\pi n t}\,dt, \quad n = \pm 1, \pm 2, \ldots$$

Here we have the reconstruction formula:

$$F_c^{-1} : f(t) = \sum_{-\infty}^{\infty} c_n\, e^{i2n\pi t}.$$

Remark The complex form in Ex 3 can be derived from the formulas in Ex1 and (Euler's) formulas

$$\begin{cases} \sin t = \dfrac{e^{it} - e^{-it}}{2i} \\ \cos t = \dfrac{e^{it} + e^{-it}}{2} \end{cases}.$$

(Note that $e^{it} = \cos t + i \sin t$, $e^{-it} = \cos t - i \sin t$)

In fact,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi t + b_n \sin 2n\pi t =$$

$$a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i2n\pi t} + e^{-i2n\pi t}}{2} \right) + b_n \left(\frac{e^{i2n\pi t} - e^{-i2n\pi t}}{2i} \right) =$$

$$a_0 + \sum_1^\infty \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{c_n} e^{i2n\pi t} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{\bar{c}_n} e^{-in2\pi t}$$

we now simply define $c_{-n} = \bar{c}_n$ and find that

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i2n\pi t}.$$

Moreover,

<u>$n>0$</u> $$c_n = \frac{a_n}{2} - i\frac{b_n}{2} = \int_0^1 f(t)\cos 2n\pi t \, dt - i\int_0^1 f(t)\sin 2n\pi t \, dt = \int_0^1 f(t) e^{-i2\pi n t} dt$$

<u>$n=0$</u> $$c_0 = a_0$$

<u>$n<0$</u> $$c_n = \bar{c}_{-n} = \frac{a_{-n}}{2} + i\frac{b_{-n}}{2} = \int_0^1 f(t)\cos(-2n\pi t)\,dt + i\int_0^1 f(t)\sin(-2n\pi t)\,dt = \int_0^1 f(t)\cos 2n\pi t \, dt - i\int_0^1 f(t)\sin 2n\pi t \, dt = \int_0^1 f(t) e^{-i2\pi n t} dt.$$

2. The Laplace Transform

$$\mathcal{L}: \; f(t) \to F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}^{-1}: \; \mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)\, e^{st} ds$$

($\mathrm{Re}\, s > \sigma_0$ so that the integrals converge).

<u>Remark</u>: For applications those invers transforms are usually found by just using a corresponding table (see e.g. Appendix 1)

It is obvious that the Laplace Transformation is <u>linear</u> i.e

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}(f(t)) + b \mathcal{L}\{g(t)\}.$$

<u>Differentiation</u>

- $\mathcal{L}(f'(t)) = s \, \mathcal{L}\{f(t)\} - f(0)$
- $\mathcal{L}(f''(t)) = s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)$

etc.

<u>Convolution</u> $\left[f * g(t) = \int_0^t f(u)\, g(t-u)\, du \right]$

- $\mathcal{L}\{f*g\} = \mathcal{L}(f) \cdot \mathcal{L}(g)$

In fact,

$$\mathcal{L}\{f*g\} = \int_0^\infty \int_0^t f(u)\, g(t-u)\, du\, e^{-st}\, dt$$

$$= \int_0^\infty \int_u^\infty f(u) g(t-u)\, e^{-st}\, du\, dt =$$

$$\int_0^\infty f(u)\, e^{-su}\left(\int_u^\infty g(t-u)\, e^{-s(t-u)}\, dt\right) du =$$

$$\int_0^\infty f(u)\, e^{-su}\left(\int_0^\infty g(x)\, e^{-sx}\, dx\right) du = \mathcal{L}(f)\,\mathcal{L}(g)$$

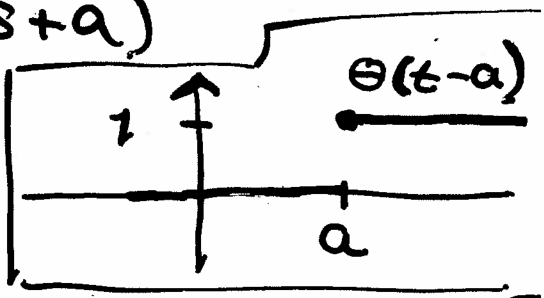
Damping

- $\mathcal{L}(e^{-at} f(t)) = F(s+a)$

In fact,

$$\mathcal{L}\{e^{-at} f(t)\}(s) = \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt =$$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}\, dt = F(s+a)$$



Timedelay

$$\mathcal{L}\{f(t-a)\,\theta(t-a)\} = e^{-as} F(s)$$

In fact

$$\mathcal{L}\{f(t-a)\,\theta(t-a)\} = \int_0^\infty f(t-a)\,\theta(t-a)\, e^{-st}\, dt =$$

$$\int_a^\infty f(t-a)\, e^{-st}\, dt = [t-a=u] = e^{-as}\int_0^\infty f(u)\, e^{-su}\, du$$

$$= e^{-as} F(s)$$

<u>Example 1</u>: $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(t) = \frac{1}{s^2}$, ... $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

In fact

$$\mathcal{L}(1) = \int_0^\infty 1\, e^{-st}\, dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

$$\mathcal{L}(t) = \int_0^\infty t\, e^{-st}\, dt = \left[\frac{-t\, e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty 1\, e^{-st}\, dt$$

$$= 0 + \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$, e.t.c.

By using this Example and the damping formula we e.g. obtain that

$$\mathcal{L}(e^{-at}) = \frac{1}{s-a}, \quad \mathcal{L}(e^{-at}t) = \frac{1}{(s-a)^2} \quad \text{etc.}$$

Example 2: $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$, $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$

In fact,

$$\mathcal{L}(e^{iat}) = \mathcal{L}(\cos at + i\sin at) = \underline{\mathcal{L}(\cos at)} + i\,\underline{\mathcal{L}(\sin at)}$$

$$\mathcal{L}(e^{iat}) = \int_0^\infty e^{iat} e^{-st} = \int_0^\infty e^{(ia-s)t}\,dt =$$

$$\left[\frac{e^{(ia-s)t}}{ia-s}\right]_0^\infty = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \underline{\frac{s}{s^2+a^2}} + i\,\underline{\frac{a}{s^2+a^2}}.$$

Example 3: Solve

$$\begin{cases} y'' + y = 1 \\ y(0) = y'(0) = 0 \end{cases}$$

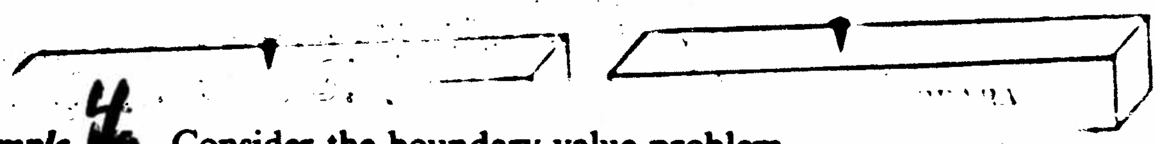
Sol: Put $\mathcal{L}\{y(t)\} = Y(s)$. Then $\mathcal{L}(y''(t)) = s^2 Y(s)$ and by Laplace transformation we get

$$s^2 Y(s) + Y(s) = \frac{1}{s};$$

$$Y(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{1+s^2}$$

Thus $\quad \underline{y(t)} = 1 - \cos t.$

Fig 3.1 [illegible] system



Example 4. Consider the boundary value problem

$$
\begin{aligned}
u_t - ku_{xx} &= 0, \qquad t > 0, \quad x > 0 \\
u(x,0) &= 0, \qquad x > 0 \\
u(0,t) &= 1, \qquad t > 0 \\
u \quad &\text{bounded}
\end{aligned}
$$

Taking the Laplace transform of both sides of the partial differential equation, that is multiplying by e^{-st} and integrating with respect to t from 0 to ∞, we obtain

$$-u(x,0) + sU(x,s) - kU_{xx}(x,s) = 0$$

or

$$U_{xx}(x,s) - (s/k)U(x,s) = 0$$

Solving this ordinary differential equation with s as a parameter we obtain

$$U(x,s) = Ae^{-\sqrt{s/k}\,x} + Be^{\sqrt{s/k}\,x}$$

where A and B are functions of s. Since bounded solutions are sought we set $B = 0$ to discard the growing exponential term and we obtain

$$U(x,s) = Ae^{-\sqrt{s/k}\,x}$$

But from the boundary condition

$$U(0,s) = \int_0^\infty u(0,t)e^{-st}\,dt = \frac{1}{s}$$

Therefore

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s/k}\,x}$$

Consulting Table 4.1 we find that the solution is

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$\boxed{\begin{aligned} \operatorname{erfc}(t) &= 1 - \operatorname{erf}(t) \\ \operatorname{erf}(t) &= \frac{2}{\sqrt{\pi}}\int_0^t e^{-\xi^2}\,d\xi \end{aligned}}$$

3. The Fourier transform

$$F: \; f(t) \to \hat{f}(\omega) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$F^{-1}: \; F^{-1}(\hat{f}(\omega)) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

(we assume that the integrals converges)

Remark 1 Still we can intepriete the formula how to reconstuct the signal $f(t)$ as a "sum" of waves (basis functions) $e^{i\omega t}$ with amplitudes $\hat{f}(\omega)$.

Remark 2 For applications those inverse transforms are usually found by just using some suitable table (see Appendix 2)

In a similar way as for Laplace-transforms we can derive a number of calculation rules for the Fouriertransform e.g. the following:

- $F\{af(t) + b\,g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$ (linearity)
- $F\{f'(t)\} = i\omega\, F\{f(t)\}$; $F\{f''(t)\} = (i\omega)^2\, \mathcal{L}\{f(t)\}$, etc (differentiation)
- $F\{f * g\} = F\{f\} \cdot F\{g\}$ (convolution) $\left[f*g(t) = \int^{\infty} f(t-u) g(u) du \right]$

- $F\{e^{iat} f(t)\} = \hat{f}(\omega - a)$
  (change in frequency)
- $F\{f(t-a)\} = e^{-i\omega a}\hat{f}(\omega)$
  (delay in time)

Example 5: $F[\theta(t)e^{-t}] = \frac{1}{1+i\omega}$

In fact,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \theta(t) e^{-t} e^{-i\omega t}\,dt = \int_{0}^{\infty} e^{-(1+i\omega)t}\,dt$$

$$= \left[\frac{-e^{-(1+i\omega)t}}{1+i\omega}\right]_0^{\infty} = \frac{1}{1+i\omega}.$$

Example 6: Solve

$$u'_t - k u''_{xx} = 0, \quad -\infty < x < \infty,\ t > 0$$

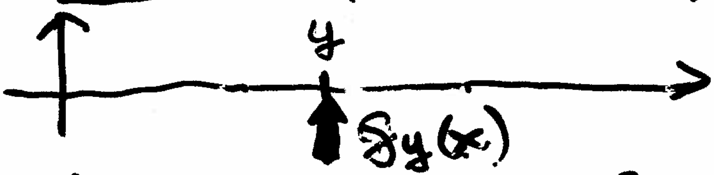
$$u(x,0) = f(x)$$

Sol: By using the Fourier transform in a similar way as we used the Laplacetransform in Example 4 we find after some calculations that

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4kt}\, d\xi$$

Remark: $G(y,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-y^2/4kt}$

is the so called Green function or unit impulsanswer of this problem:



$$\begin{cases} G'_t - k G''_{xx} = 0 \\ G(x,0) = \delta_y(x) \end{cases}$$

Green's method: Solution $u = f * G$

Sampling

$S: f(t) \longrightarrow \{f(n\delta)\}$, δ = length of sampling-interval

The (famous) sampling theorem:

A continuous bandlimited signal $f(t)$ (i.e. a continous signal with Fourier-transform $\hat{f}(\omega) = 0$ for $|\omega| \geq C$) kan be uniquely evaluated from the values in equivadistant points if the length of the sampling interval is at most π/C. Then we have

$$S^{-1}: \; f(t) = \sum_{-\infty}^{\infty} f\left(\frac{k\pi}{c}\right) \frac{\sin(ct - k\pi)}{ct - k\pi}.$$

Remark In this connction we also want to mention two discrete transforms

- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)

These transforms are very useful for severel applications but in this short introduction of transform methods we must only refer the reader to the literature.

4. The z-transform

Consider timediscrete signals

$\{x_n\}_0^\infty = \{x_0, x_1, x_2, \dots \}$ or

$\{x_n\}_{-\infty}^{\infty} = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2 \dots\}$.

The notation

$$\{1, 2, \overset{\downarrow}{5}, 6, -1, \dots\}$$

means that $x_0 = 5$.

$$Z: \{x_n\} \longrightarrow X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

$$Z^{-1}: Z^{-1}[X] = \{x_n\}_0^\infty$$

The Z-transform may be regarded as a discrete version of the Laplace-transform so it is not surprising that similar calculation rules holds. e.g the following:

- $Z[a\{x_n\} + b\{y_n\}] = aZ[\{x_n\}] + bZ[\{y_n\}]$
  (linearity)
- $Z[\{a^n x_n\}] = X\left(\frac{z}{a}\right)$
  ("damping")
- $Z[\{x_n\} * \{y_n\}] = Z[\{x_n\}] \cdot Z[\{y_n\}]$
  (convolution)

$$\left[\{x_n\} * \{y_n\} = \left\{\sum_{k=1}^{n} x_{n-k} y_k\right\}_{n=0}^{\infty}\right]$$

- $X'(z) = Z[\{0, 0, -x_1, -2x_2, \dots\}]$
(derivation)

$Z[\{0, x_0, x_1, \dots\}] = z^{-1} X(z)$

$Z[\{0, 0, x_0, x_1, \dots\}] = z^{-2} X(z)$

etc. (forward shift)

$Z[\{x_0, \overset{\downarrow}{x_1}, x_2, \dots\}] = z X(z) - x_0 z$

$Z[\{x_0, x_1, \overset{\downarrow}{x_2}, \dots\}] = z^2 X(z) - x_0 z^2 - x_1 z$

etc. (backward shift)

In fact,

$Z[\{x_0, \overset{\downarrow}{x_1}, x_2, \dots\}] = x_1 + x_2 \frac{1}{z} + x_3 \frac{1}{z^2} + \dots$

$= x_0 z + x_1 + x_2 \frac{1}{z} + x_3 \frac{1}{z^2} + \dots - x_0 z =$

$z\left(x_0 + x_1 \frac{1}{z} + x_2 \frac{1}{z^2} + x_3 \frac{1}{z^3} + \dots\right) - x_0 z = z X(z) - x_0 z$

etc.

Example 7:

a) Unit step sequence $\{0, 0, \overset{\downarrow}{1}, 1, 1 \dots\} = \{\theta_n\}$

$Z[\{\theta_n\}] = 1 + \frac{1}{z} + \frac{1}{z^2} + \dots = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$ $\boxed{|z| > 1}$

b) Unit impulse sequence $\{\dots 0, 0, \overset{\downarrow}{1}, 0, 0, \dots\} = \{\delta_n\}$

e.g. $\{\dots 0, 0, \overset{\downarrow}{0}, 0, 1, 0, 0 \dots\} = \{\delta_{n-2}\}$

$Z[\{\delta_n\}] = 1$

$Z[\{\delta_{n-2}\}] = \frac{1}{z^2}$

etc.

c) Unit ramp sequence $\{\cdots 0,0,\overset{\downarrow}{0},1,2,\ldots\} = \{r_n\}$

$$Z[\{r_n\}] = \frac{z}{(z-1)^2}$$

In fact

$$Z[\{\cdots 0,0,0,1,2,\ldots\}] = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \quad := f(z)$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \qquad (|z|<1)$$

$$\therefore \frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots\cdots \qquad \text{(differentiation)}$$

$$z \cdot \frac{1}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots\cdots$$

put $\frac{1}{z}$ in this formula instead of z and we get the desired

$$f(z) = \frac{1}{z} + 2\frac{1}{z^2} + \frac{3}{z^3} + \cdots = \frac{1}{z}\,\frac{1}{\left(1-\frac{1}{z}\right)^2} = \underline{\frac{z}{(z-1)^2}} \quad (|z|>1)$$

Remark: The Z-transform is very useful for solving <u>difference equations</u> and for handling discrete linear systems

Remark: More examples of useful transform couples and calculation rules can be found in our Appendix 3.

5. Wavelet - Transforms

■ The wavelets idea is fairly new but has already been proven to be much more efficient than many other transforms e.g. for some applications in

a) Signal processing,

b) Image processing.

In this case the history start with what is now called the

<u>motherwelet</u> ψ.

Typically the function ψ has the following properties

(*) $\int_{-\infty}^{\infty} \psi(t)\,dt = 0$,

(**) ψ is well localized in <u>both</u> time and frequence,

and ψ also satisfies some additional technical conditions.

It can then be proved that the system

$$\{\psi_{j,k}(t)\}_{j,k=-\infty}^{\infty},$$

where

$$\psi_{j,k}(t) := 2^{j/k}\psi(2^j t-k)$$

are just translations, dilations and normalizations of the original mother wavelet, is a (complete) orthonormal basis and a signal $f(t)$ can be reconstructed by using the usual (generalized) Fourier idea:

$$W^{-1}: f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k}(t)$$

[Here, of course,

$W: f(t) \to \{\langle f, \psi_{j,k}\rangle\}_{j,k=-\infty}^{\infty}$]

Remark 1: In Appendix 4 we have included a motivation and illustrations, which makes it easier to understand the terminology och formulas above. The motivation is done via a natural approximation procedure and with ~~the~~ classical Haar wavelet as the mother wavelet.

Remark 2: The transform above corresponds to the Fourier series Transform but there also exists a similar integral transform of Fourier Transform type.

Remark 3: The Wavelets Transforms are not very useful for handcalculation but nowadays there are programs easily avaible which makes them very powerful for some applications You can reach such information on the following www addresses:

- http://www.wavelet.org/
(Wavelet Digest + seach engine + links + ..)
- http://www.finah.com/
(Many practical examples)
- http://www.tyche.mat.univie.ac.at/Gabor/index.html
(Gabor analysis)
- http://www.sm.luth.se/~grip
(e.g. the Licentiate and PhD thesis of Niklas Grip contain such information)

■ Some groups in Sweden working with wavelets and applications (also industrial)

- KTH: Jan-Olov Strömberg (janolov@math.kth.se)
- Chalmers: Göran Bergh (bergh@math.chalmers.se)
- LTU: Lars-Erik Persson (larserik@sm.luth.se)
(Uppsala)

Some Books on wavelets

[1] J. Bergh, F. Ekstedt and M. Lindberg Wavelets, Studentlitteratur, 1999.

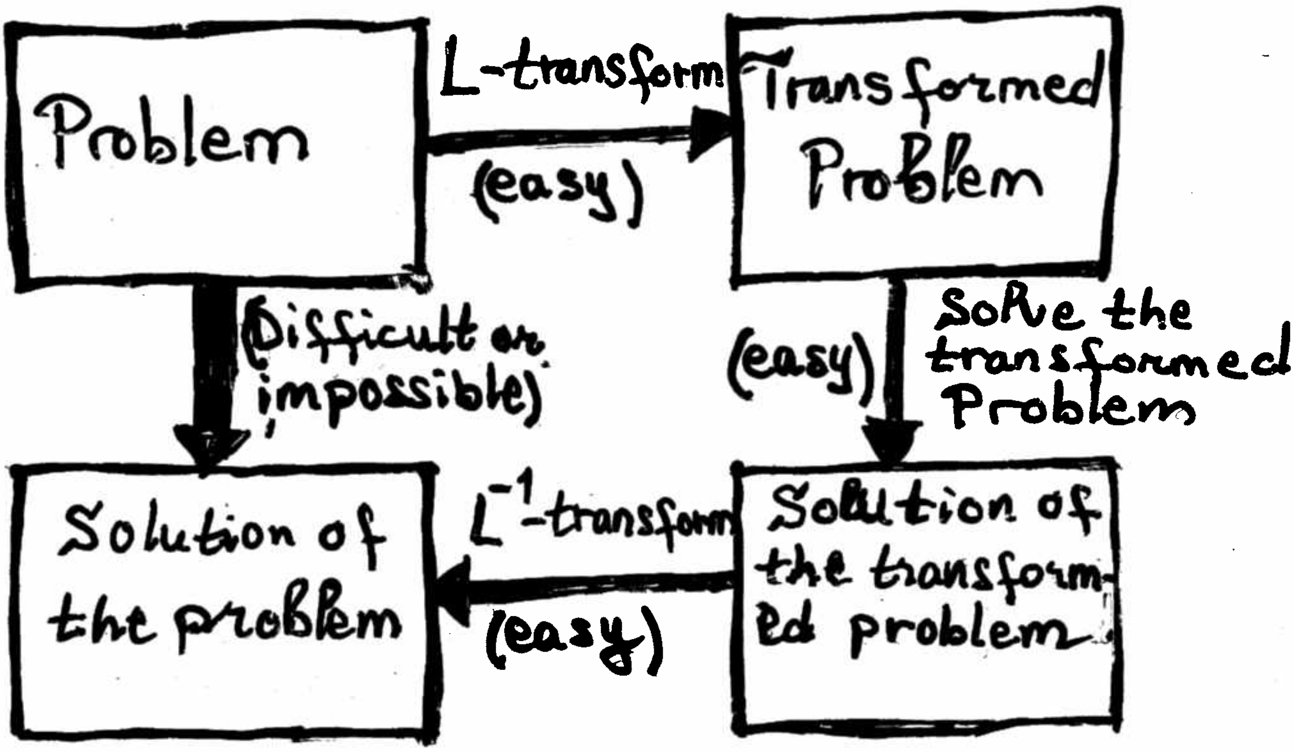
[2] S. G. Mallat, A Wavelet Tour of Signal Processing, Academic Press, 1999.

[3] C.S Burrus, R.A Gopinath and H. Guo, Introduction to Wavelets and Wavelet Transforms, A Primar, Prentice Hall, 1998.

[4] K. Gröchenig Foundations of Time-Frequency Analysis, Birkhäuser, 2000.

6. The general Transform idea

$f(t); \{a_n\} \longrightarrow \tilde{f}(s); \{c_n\}$



- The key problem is to choose a suitable transform for the problem at hand.

In this lecture we have presented some useful transforms but there exist a lot of other transforms in the literature.
In Appendix 5 we have presented additional transforms taken from the book:
[*] L. Debnath, Integral Transforms and Their Applications, CRS Press, 1995.
In almost each case also the inverse transform is pointed out explicitely and the corresponding useful tables are included. In each of the cases concrete problems are solved

7. Continuous linear systems

Many linear systems e.g. for technical applications can be described with a linear differential equation

$$(*) \quad a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots a_0 y_0(t) =$$
$$b_k x^k(t) + b_{k-1} x^{k-1}(t) + \cdots b_0 x(t)$$

We assume also that $y(0) = y'(0) = \cdots y^{(n)}(0) = 0$. Laplace transformation of $(*)$ now gives (with $Y(s) = \mathcal{L}(y(t)), \mathcal{L}(x(t)) = X(s)$)

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots a_0) Y(s) = (b_k s^k + b_{k-1} s^{k-1} + \cdots + b_0) X(s)$$

and we get

$$\frac{Y(s)}{X(s)} = \frac{b_k s^k + b_{k-1} s^{k-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

Transfer function $H(s)$

$$\boxed{H(s) = \frac{b_k s^k + b_{k-1} s^{k-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}}$$

So for each insignal (with $X(s)$) we get the corresponding solution

$$Y(s) = H(s) \cdot X(s) \text{ i.e.}$$

$$y(t) = h(t) * x(t)$$

How can we find $H(s)$?

For a unit impuls $\delta(t)$ we have

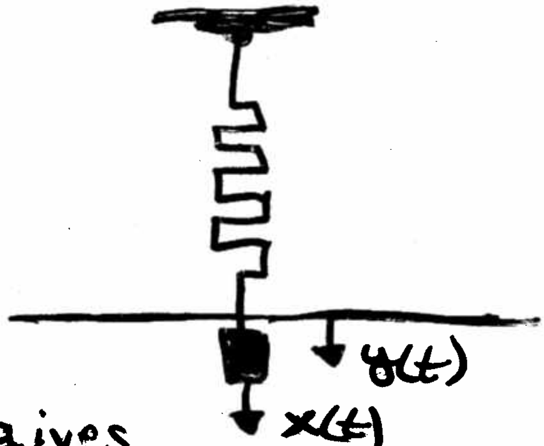
$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = e^0 = 1$$

This means that the system answers on the unit impuls in the following way:

$$\begin{cases} y(t) = h(t) * \delta(t) = h(t) \\ Y(s) = H(s) \end{cases}$$

In technical applications $h(t)$ are many times called the unit impulse answer.

Example 8:



$m\ddot{y}(t) + c\dot{y}(t) + ky(t)$
$= c\dot{x}(t) + ax(t)$

Laplacetransformation gives

$$(ms^2 + cs + k)\,Y(s) = (cs + a)\,X(s)$$

i.e. the transfer function is

$$H(s) = \frac{cs + a}{ms^2 + cs + k}$$

For example if the insignal $x(t) = \sin \omega t$ so that $X(s) = \frac{\omega}{s^2 + \omega^2}$ we have that (the laplacetransform of the outsignal) is

$$Y(s) = H(s)\,X(s) = \frac{cs + a}{ms^2 + cs + k} \cdot \frac{\omega}{s^2 + \omega^2}$$

E.g. with $m = 1$ kg, $c = 0$, $k = a = 1000$ N/m and $\omega = 2\pi$. Then

$$Y(s) = \frac{1000}{s^2 + 1000} \cdot \frac{2\pi}{s^2 + 4\pi^2} = \frac{D}{s^2 + \pi^2} - \frac{D}{s^2 + 1000},$$

where $D = 2000\pi/(1000 - 4\pi^2)$.

Thus, $y(t) = \frac{D}{2\pi} \sin 2\pi t - \frac{D}{\sqrt{1000}} \sin \sqrt{1000}\, t$

$$\approx 1.04 \sin 6.28t - 0.207 \sin 31.6\, t \quad \blacksquare$$

It is sometimes ~~also~~ interesting to find the unit step answer, i.e. the system's reaction of the insignal

$\theta(t) = \{1, t > 0$

We know that $\mathcal{L}(\varepsilon(t)) = \frac{1}{s}$ so that

$$Y(s) = H(s)\,\frac{1}{s}$$

Example 9: A system has the transfer function

$$H(s) = \frac{3}{(s+1)(s+3)}$$

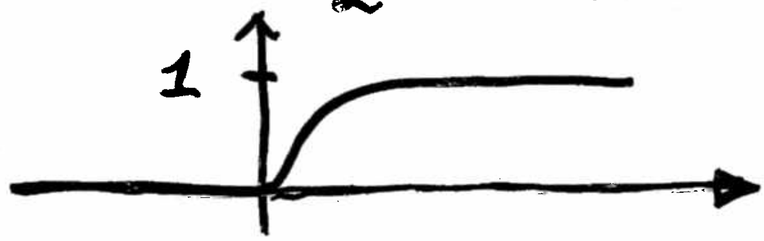
Calculate the unit step answer!

We have

$$Y(s) = H(s)\,E(s) = \frac{3}{(s+1)(s+3)} \cdot \frac{1}{s} =$$

$$\frac{1}{s} - \frac{3}{2(s+1)} + \frac{1}{2(s+3)}$$

Thus $y(t) = 1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t}$



8. Discrete linear systems

A discrete Linear system can be described as follows

$$(*) \quad a_0 y_n + a_1 y_{n-1} + \dots + a_m y_{n-m} = b_0 x_n + b_1 x_{n-1} + \dots + b_k x_{n-k}$$

(Formally $\{a_n\} * \{y_n\} = \{b_n\} * \{x_n\}$)

we put $Y(z) = Z[\{y_n\}]$, $X(z) = Z[\{x_n\}]$, and make Z-transformation of $(*)$ and get

$$\left(a_0 + a_1 \frac{1}{z} + \dots + a_m \frac{1}{z^m}\right) Y(z) = \left(b_0 + b_1 \frac{1}{z} + \dots + b_k \frac{1}{z^k}\right) X(z)$$

i.e.

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 \frac{1}{z} + \dots + b_k \frac{1}{z^k}}{a_0 + a_1 \frac{1}{z} + \dots + a_m \frac{1}{z^m}}$$

$$\boxed{\begin{array}{l} \text{Transferfunction } H(z) \\ H(z) = \dfrac{b_0 + b_1 \frac{1}{z} + \dots + b_k \frac{1}{z^k}}{a_0 + a_1 \frac{1}{z} + \dots + a_m \frac{1}{z^m}} \end{array}}$$

So, for each insignal (with $X(z)$) we get the solution

$$Y(z) = H(z) X(z) \text{ i.e}$$

$$\{y_n\} = \{h_n\} * \{x_n\}$$

How can we find $H(z)$?

For the unit impulse $\{\delta_n\}$ we have

$$Z[\{\delta_n\}] = 1 + 0 \cdot \frac{1}{z} + \cdots = 1$$

This means that the system answers on the unit impuls in the following way:

$$\{y_n\} = \{h_n\} * \{\delta_n\} = \{h_n\}$$

$$Y(z) = H(z)$$

In technical applications $\{h_n\}$ is called the <u>unit impulse answer</u>.

Example 10: A technical linear discrete system has the transfer function $H(z) = \frac{1}{z+0.8}$. Calculate the unit step answer!

Solution: $\{\sigma_n\} = \{1, 1, 1, \dots\}$. We have

$$X(z) = Z[\{\sigma_n\}] = \frac{z}{z-1}$$

so that

$$y(z) = \frac{z}{(z-1)(z+0.8)} = z\,\frac{5}{9}\left[\frac{1}{z-1} - \frac{1}{z+0.8}\right]$$

Inverstransformation gives:

$$\{y_n\} \text{ with } y_n = \frac{5}{9}\left(1 - (-0.8)^n\right).$$

Some Swedish Teaching Books in connection to continous and discrete linear systems:

[*] L. Bergström, B. Snaar, Laplacetransformer och z-transformer, Natura läromedel, 1997.

[*] H. Lennerstad, C. Jogréus, Serier och transformer, Studentlitteratur 1999.

[*] H. Sollervall, B. Styf, Transformteori för ingenjörer, Bokförlaget KUB 1997.

[*] S. Sparre, Föreläsningar i linjära system, KF-Sigmatryck, 1995.

[*] I. Stridh, Transformteori för TNDE23, Linköpings Tekniska Högskola, 1997.

9. Further Examples

Example 11: Calculate $\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx$ for $a>0$.

Sol: we consider

$$f(t)=\int_0^\infty \frac{\sin tx}{x(1+x^2)}dx,\ t>0$$

and its Laplace transform

$$\mathcal{L}(f(t))=\tilde{f}(s)=\int_0^\infty\left(\int_0^\infty \frac{\sin tx}{x(1+x^2)}dx\right)e^{-st}dt$$

$$=\int_0^\infty \underbrace{\left(\int_0^\infty \sin tx\, e^{-st}dt\right)}_{\mathcal{L}(\sin tx)}\frac{1}{x(1+x^2)}dx =$$

$$\int_0^\infty \frac{x}{x^2+s^2}\,\frac{1}{x(1+x^2)}dx=\frac{1}{s^2-1}\int_0^\infty\left(\frac{1}{1+x^2}-\frac{1}{x^2+s^2}\right)dx$$

$$=\frac{1}{s^2-1}\left(\frac{\pi}{2}-\frac{\pi}{2}\frac{1}{s}\right)=\frac{\pi}{2}\frac{s-1}{(s-1)(s+1)\cdot s}=$$

$$\frac{\pi}{2}\frac{1}{s(s+1)}=\frac{\pi}{2}\left(\frac{1}{s}-\frac{1}{s+1}\right)$$

Thus, by inversed transformation,

$$f(t)=\frac{\pi}{2}(1-e^{-t})$$

so we conclude

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx=\frac{\pi}{2}(1-e^{-a}),\ a>0.$$

Example 12 (Dirichlet's problem for a halfplane) Solve

$$(*)\quad \begin{cases} u''_{xx}+u''_{yy}=0, & -\infty<x<\infty,\ y\geq 0 \\ u(x,0)=f(x) \\ u(x,y)\to 0 \text{ as } |x|\to\infty,\ y\to\infty \end{cases}$$

Sol: We make Fouriertransformation with respect to x and get

$$U=U(\omega,y)=\int_{-\infty}^{\infty}u(x,y)\,e^{-i\omega x}\,dx$$

so that $(*)$ now transforms to

$$\begin{cases} \dfrac{d^2U}{dy^2}-\omega^2U=0, \\ U(\omega,0)=\hat{f}(\omega), \\ U(\omega,y)\to 0 \text{ as } y\to\infty. \end{cases}$$

The solution of this transformed system is $U(\omega,y)=\hat{f}(\omega)\,e^{-|\omega|y}$.

We apply the convolution theorem and find that

$$u(x,y)=\int_{-\infty}^{\infty}f(\xi)\,g_y(x-\xi)\,d\xi,$$

where $g_y(x)$ is the inverse Fourier transform of $e^{-i\omega y}$ i.e.

$$g_y(x)=\frac{1}{\pi}\,\frac{y}{x^2+y^2}.$$

Consequently the desired solution is

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}f(\xi)\,\frac{1}{(x-\xi)^2+y^2}\,d\xi,\quad y>0.$$

(This is famous Poisson's integral formula)

Exercises Lecture 6

Exercise 1: a) Find the inverse Laplacetransform to

$$F(s) = e^{-2s} \frac{1}{s^2+8s+15}.$$

$$\boxed{f(t) = \left(\frac{1}{2} e^{-3(t-2)} - \frac{1}{2} e^{-5(t-2)}\right)\theta(t}$$

b) Find the unitstepanswer to a system with the transferfunction

$$H(s) = \frac{3}{(s+1)(s+3)}.$$

$$\boxed{y(t) = 1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3}}$$

Exercise 2*: Use the Laplacetransform to solve:

$$\begin{cases} u'_t = u''_{xx}, & 0 \leq x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 1, & t > 0, \\ u(x,0) = 1 + \sin \pi x, & 0 < x < 1, \end{cases}$$

$$\boxed{u = 1 + e^{-\pi^2 t} \sin \pi x}$$

Exercise 3*: a) Derive the convolution formula $F\{f * g\} = F\{f\} \cdot F\{g\}$ for Fourier transforms

b) Beräkna Fouriertransformen av signalen $f(t) = \theta(t-3)\, e^{-(t-3)}$.

$$\boxed{\hat{f}(w) = e^{-3iw} \frac{1}{1+iw}}$$

Exercise 4: a) Define the Haar scaling function φ and the Haar wavelet function ψ.

b) Illustrate $\psi(t-2)$, $\psi(4t)$, $\psi(4t-1)$, $\psi(4t-3)$ and $2\psi(4t-2)$ in a ty-plane

c) Explain how a signal $f(t)$ can be represented by using a system of basis functions obtained by making dilations, translations and normalizations of the Haar wavelet.

Exercise 5: Solve with Laplacetransformation the following system of differentialequations

$$\begin{cases} x' - 2x + 3y = 0 \\ y' - y + 2x = 0 \end{cases} \qquad \begin{cases} x(0) = 8 \\ y(0) = 3 \end{cases}$$

$$\boxed{\begin{array}{l} x(t) = 5e^{-t} + 3e^{4t} \quad \left(X(s) = \frac{5}{s+1} + \frac{3}{s-4} \right) \\ y(t) = 5e^{-t} - 2e^{4t} \quad \left(Y(s) = \frac{5}{s+1} - \frac{2}{s-4} \right) \end{array}}$$

*1 **Exercise 6:** A system has the transfer function $H(s) = 1/(1+sT)$ and we insert the insignal $x(t) = \sin \omega t$.
Calculate the outsignal $y(t)$.

$\boxed{\text{one of these}}$

$$\boxed{y(t) = \frac{1}{1+\omega^2 T^2} \omega T e^{-t/T} + \sin \omega t - \omega T \cos \omega t}$$

*2 **Exercise 7:** A technical discrete linear system has the transfer function $H(z) = \frac{1}{2z+1}$. Calculate the unit step answer.

*3 **Exercise 8:** A technical discrete linear system has the unit impulse answer $\{0.7^n\}$. Calculate the outsignal when the insignal $= \{a^n\}$ $a \neq 0.7$

$\{\frac{10}{[illegible]}(a^{n+1} - 0.7^n)\sigma_n\}$

TABLE 4.1. Laplace Transforms

$f(t)$	$F(s)$
1	$s^{-1}, \quad s > 0$
$\exp(at)$	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n$ a positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin at$ and $\cos at$	$\dfrac{a}{s^2+a^2}$ and $\dfrac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$ and $\cosh at$	$\dfrac{a}{s^2-a^2}$ and $\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n\exp(at)$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$H(t-a)$	$s^{-1}\exp(-as), \quad s > 0$
$\delta(t-a)$	$\exp(-as)$
$H(t-a)f(t-a)$	$F(s)\exp(-as)$
$\operatorname{erf}\sqrt{t}$	$s^{-1}(1+s)^{-1/2}, \quad s > 0$
$\dfrac{1}{\sqrt{t}}\exp\dfrac{-a^2}{4t}$	$\sqrt{\pi/s}\,\exp(-a\sqrt{s}), \quad (s > 0)$
$\operatorname{erfc}\dfrac{a}{2\sqrt{t}}$	$s^{-1}\exp(-a\sqrt{s}), \quad s > 0$
$\dfrac{a}{2t^{3/2}}\exp\dfrac{-a^2}{4t}$	$\sqrt{\pi}\,\exp(-a\sqrt{s}), \quad s > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

Appendix 2

4.8 Formelsamling

Räkneregel (sid.)	$\mathbf{f(t)}$	$\widehat{\mathbf{f}}(\omega)$	
Fourierintegral (135)	$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$F1$
Inversformel (138)	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega t}d\omega$	$\widehat{f}(\omega)$	$F2$
Linearitet (141)	$af(t)+bg(t)$	$a\widehat{f}(\omega)+b\widehat{g}(\omega)$	$F3$
Skalning (143)	$f(at) \quad (a \neq 0)$	$\frac{1}{\|a\|}\widehat{f}(\frac{\omega}{a})$	$F4$
Teckenbyte (144)	$f(-t)$	$\widehat{f}(-\omega)$	$F5$
Komplexkonjugat (145)	$\overline{f(t)}$	$\overline{\widehat{f}(-\omega)}$	$F6$
Tidsförskjutning (145)	$f(t-T)$	$e^{-i\omega T}\widehat{f}(\omega)$	$F7$
Fr.förskjutning (146)	$e^{i\Omega t}f(t)$	$\widehat{f}(\omega-\Omega)$	$F8$
Ampl.modulering (147)	$f(t)\cos\Omega t$	$\frac{1}{2}(\widehat{f}(\omega-\Omega)+\widehat{f}(\omega+\Omega))$	$F9a$
Ampl.modulering (147)	$f(t)\sin\Omega t$	$\frac{1}{2i}(\widehat{f}(\omega-\Omega)-\widehat{f}(\omega+\Omega))$	$F9b$
Symmetri (148)	$\widehat{f}(t)$	$2\pi f(-\omega)$	$F10$
Tidsderivering (148)	$f'(t)$	$i\omega\widehat{f}(\omega)$	$F11$
Frekv.derivering (149)	$(-it)f(t)$	$\widehat{f}'(\omega)$	$F12$
Tidsfaltning (149)	$f(t)*g(t)$	$\widehat{f}(\omega)\widehat{g}(\omega)$	$F13$
Frekv.faltning (151)	$f(t)g(t)$	$\frac{1}{2\pi}\widehat{f}(\omega)*\widehat{g}(\omega)$	$F14$

Transformpar (sid.)

Deltafunktion (152)	$\delta(t)$	1	$F15$
derivata av (152)	$\delta^{(n)}(t)$	$(i\omega)^n$	$F16$
Exponential (152)	$\theta(t)e^{-at}$	$\frac{1}{a+i\omega} \quad (a>0)$	$F17$
Exponential (152)	$(1-\theta(t))e^{at}$	$\frac{1}{a-i\omega} \quad (a>0)$	$F18$
Exponential (153)	$e^{-a\|t\|} \quad (a>0)$	$\frac{2a}{a^2+\omega^2}$	$F19$
Stegfunktion (153)	$\theta(t)$	$\pi\delta(\omega)+\frac{1}{i\omega}$	$F20$
Konstant (153)	1	$2\pi\delta(\omega)$	$F21$
(154)	$\frac{\sin\Omega t}{\pi t}$	$\theta(\omega+\Omega)-\theta(\omega-\Omega)$	$F22$
Gaussfunktion (155)	$\frac{1}{\sqrt{4\pi A}}e^{-t^2/(4A)}$	$e^{-A\omega^2} \quad (A>0)$	$F23$

Appendix 3

6.9 Formelsamling

Räkneregel (sid.)	$\{x_n\}_{n=0}^{\infty}$	$X(z)$	
Definition (211)	x_n	$\sum_{n=1}^{\infty} x_n z^{-n}$	Z1
Linearitet (216)	$a\{x_n\} + b\{y_n\}$	$aZ[\{x_n\}] + bZ[\{y_n\}]$	Z2
Dämpning (216)	$a^n x_n$	$X(\frac{z}{a})$	Z3
(217)	nx_n	$-zX'(z)$	Z4
Derivering (217)	$(1-n)x_{n-1}\sigma_{n-1}$	$X'(z)$	Z5
Faltning (213)	$\{x_n\} * \{y_n\}$	$X(z)Y(z)$	Z6
Förskjutning framåt (219)	$x_{n-k}\sigma_{n-k}, (k \geq 0)$	$z^{-k}X(z)$	Z7
Förskjutning bakåt (220)	$x_{n+k}, \ (k \geq 0)$	$z^k X(z) - \sum_{j=0}^{k-1} x_j z^{k-j}$	Z8

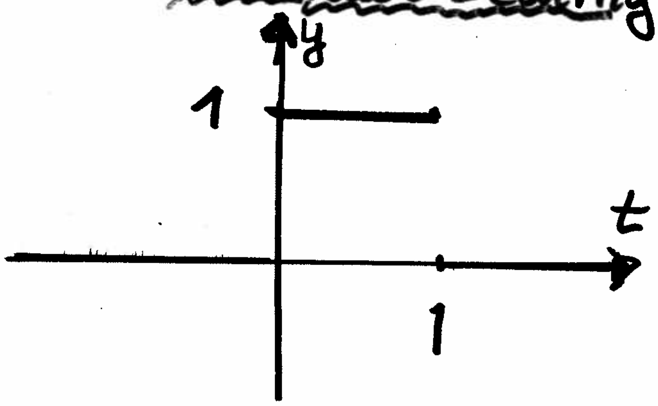
Transformpar (sid.)

Enhetssteg (213)	σ_n	$\frac{z}{z-1}$	Z9
Enhetspuls (213)	δ_n	1	Z10
fördröjd (213)	δ_{n-k}	z^{-k}	Z11
Exponential (220)	a^n	$\frac{z}{z-a}$	Z12
Rampfunktion (215)	$r_n = n\sigma_n$	$\frac{z}{(z-1)^2}$	Z13
Sinus (221)	$\sin n\theta$	$\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$	Z14
Dämpad sin (221)	$a^n \sin n\theta$	$\frac{za\sin\theta}{z^2 - 2za\cos\theta + a^2}$	Z15
Cosinus (221)	$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$	Z16
Dämpad cos (221)	$a^n \cos n\theta$	$\frac{z(z-a\cos\theta)}{z^2 - 2za\cos\theta + a^2}$	Z17

1. The Haar wavelet

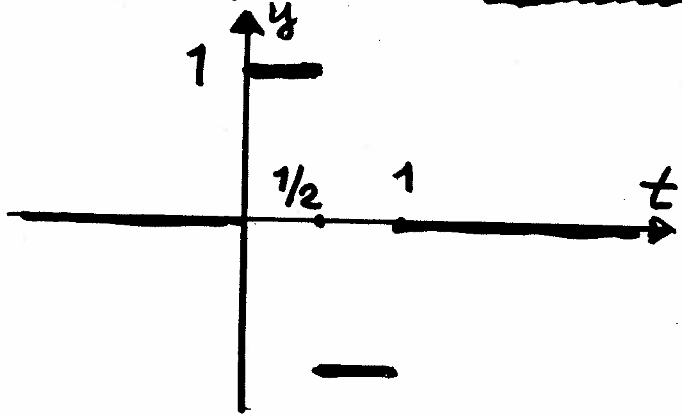
Appendix 4 (A)

The Haar scaling function



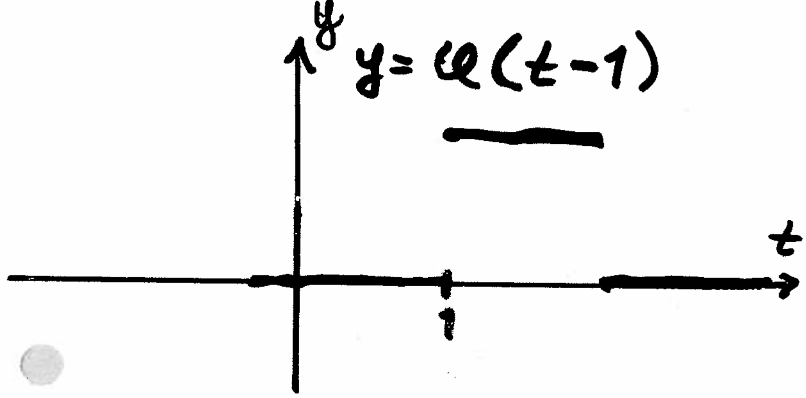
$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

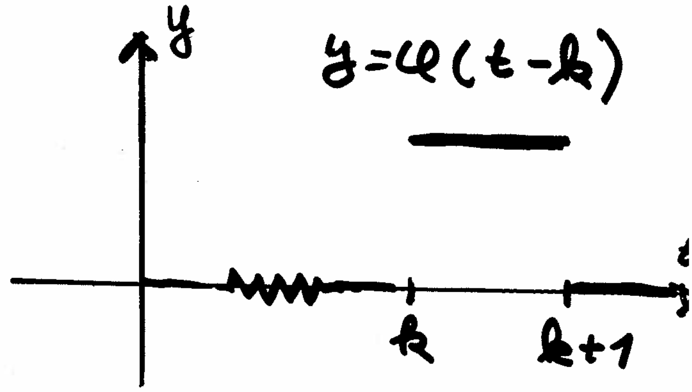
The Haar wavelet



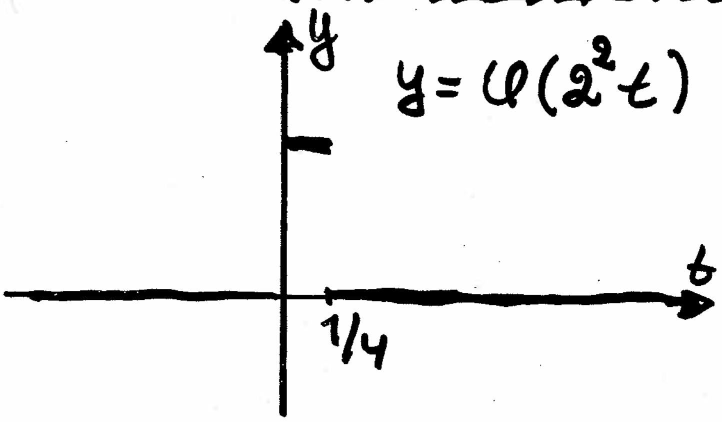
$$\psi(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ -1, & 1/2 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

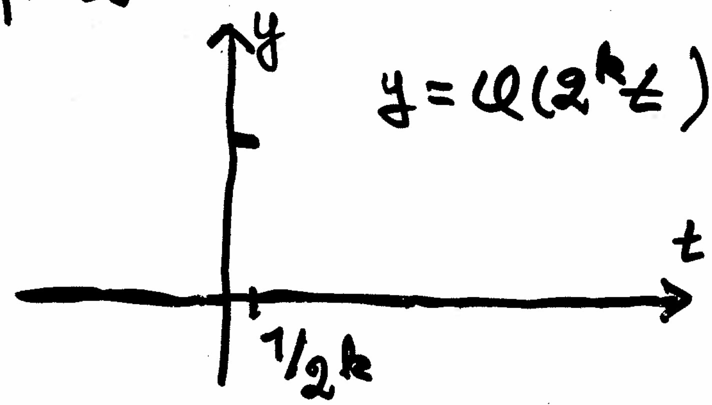
Translations of φ





Dilations of φ

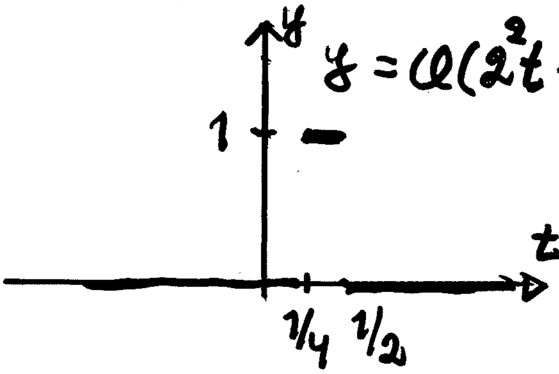




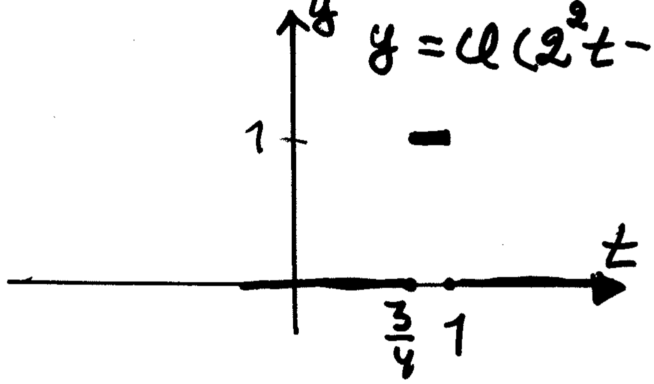
B

Dilations and translations

$y = \varphi(2^2 t - 1)$

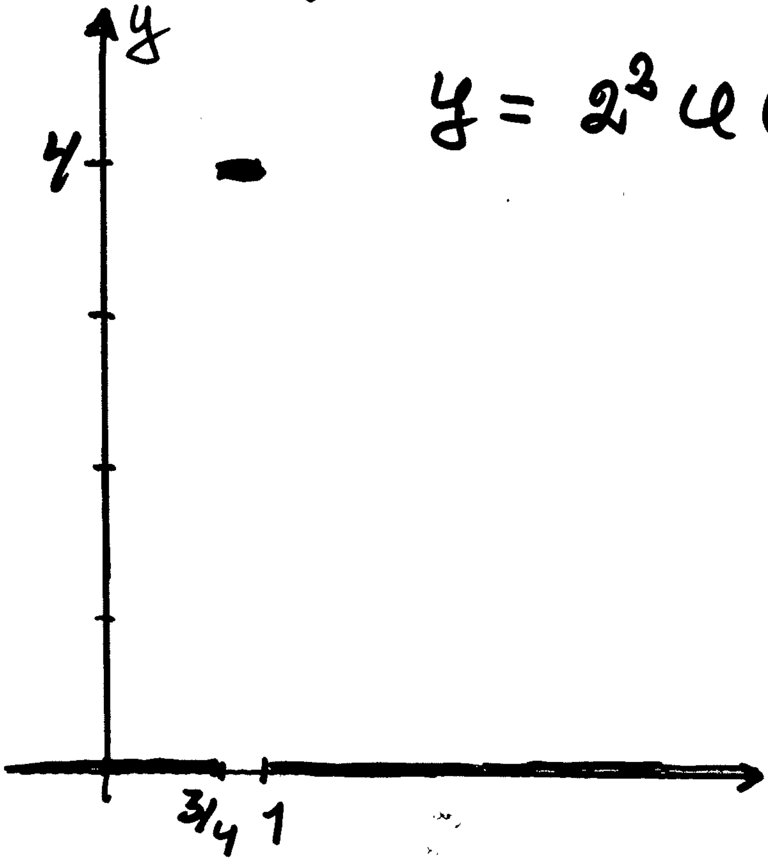


$y = \varphi(2^2 t - 3)$



Dilations, translations and normalization

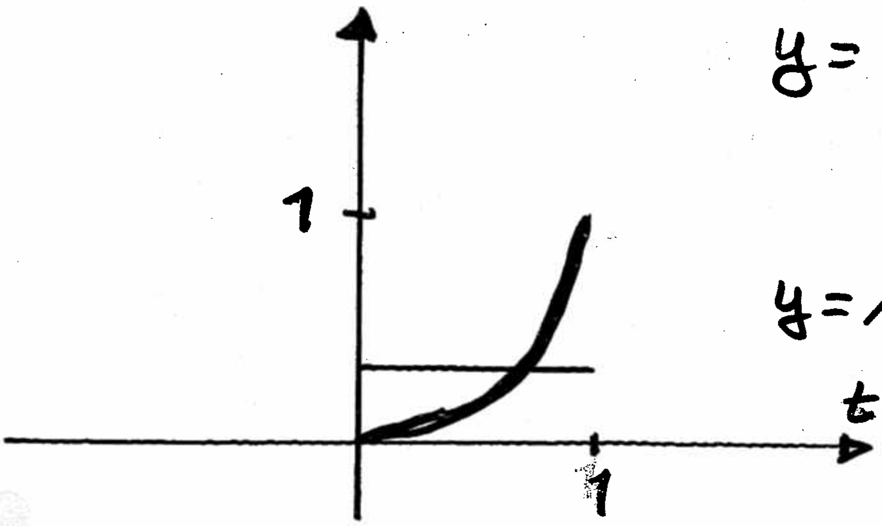
$$y = 2^2 \varphi(2^2 t - 3)$$



Remark: Note that e.g

$$\begin{cases} \varphi(t) = \varphi(2t) + \varphi(2t-1) \\ \psi(t) = \varphi(2t) - \varphi(2t-1) \end{cases}$$

9. An approximation example

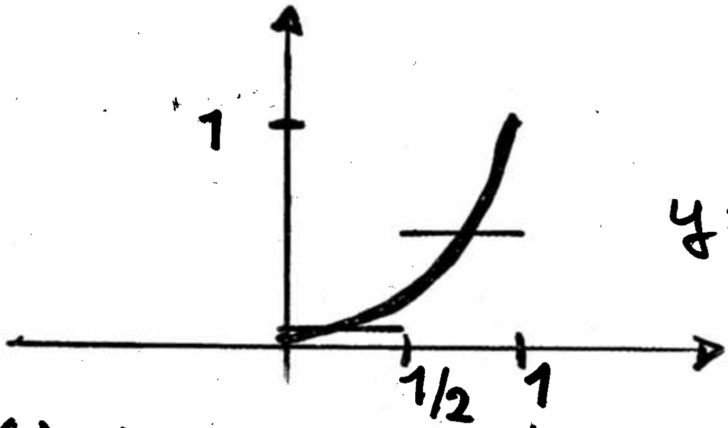


$y = f(t)$ (e.g. $f(t) = t^2$

$y = A_0(t)$

a) Approximation by the mean value

$$f(t) \approx A_0(t) = \left(\frac{1}{1}\int_0^1 f(s)\,ds\right) \varphi(t)$$



$y = A_1(t)$

b) Approximation with step function (2 steps)

$$f(t) \approx A_1(t) = 2\int_0^{1/2} f(s)\,ds\ \varphi(2t) + 2\int_{1/2}^{1} f(s)\,ds\ \varphi(2t-1)$$

$$= \underbrace{\int_0^1 f(s)\sqrt{2}\,\varphi(2s)\,ds}_{a_0}\ \underbrace{\sqrt{2}\,\varphi(2t)}_{\varphi_0} +$$

$$\underbrace{\int_0^1 f(s)\sqrt{2}\,\varphi(2s-1)\,ds}_{a_1}\ \underbrace{\sqrt{2}\,\varphi(2t-1)}_{\varphi_1}$$

D

c) Approximation with stepfunction (4 steps)

$$f(t) \approx A_2(t) = 4\int_0^{1/4} f(s)\,ds\ \varphi(4t) + 4\int_{1/4}^{1/2} f(s)\,ds\ \varphi(4t-1)$$

$$+ 4\int_{1/2}^{3/4} f(s)\,ds\ \varphi(4t-2) + 4\int_{3/4}^{1} f(s)\,ds\ \varphi(4t-3)$$

$$= \underbrace{\int_0^1 f(s)\,2\varphi(4s)\,ds}_{a_0}\ \underbrace{2\varphi(4t)}_{\varphi_0} + \underbrace{\int_0^1 f(s)\,2\varphi(4s-1)\,ds}_{a_1}\ \underbrace{2\varphi(4t-1)}_{\varphi_1}$$

$$+ \underbrace{\int_0^1 f(s)\,2\varphi(4s-2)\,ds}_{a_2}\ \underbrace{2\varphi(4t-2)}_{\varphi_2} + \underbrace{\int_0^1 f(s)\,2\varphi(4s-3)\,ds}_{a_3}\ \underbrace{2\varphi(4t-3)}_{\varphi_3}$$

c) Approximation with stepfunction (2^n steps)

$$f(t) \approx \sum_{0}^{k=2^n-1} a_k\,\varphi_k(t)$$

where

$$a_k = \int_0^1 f(s)\,2^{n/2}\varphi(2^n s-k)\,ds \quad \text{("Fourier-coefficients")}$$

and

$$\varphi_k(t) = 2^{n/2}\varphi(2^n t-k) \quad \text{("Basis functions")}$$

Some further transforms.

For more information and applications se

[*] L. Debnath, Integral Transforms and Their Applications, CRS Press, 1995

[1] Fourier Cosine Transform

$$F_c : f(t) \to \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt$$

$$F_c^{-1} : f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega t \, d\omega$$

[2] Fourier Sine Transform

$$F_s : f(t) \to \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt$$

$$F_s^{-1} : f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega t \, d\omega$$

[3] Hankel Transforms

(defined by using Bessel functions $J_n, n \geq 0$,

$$H_n : f(r) \to \hat{f}_n(y) = \int_0^\infty J_n(yr) f(r) r \, dr$$

$$H_n^{-1} : f(r) = \int_0^\infty J_n(yr) \hat{f}_n(y) y \, dr$$

[4] Mellin Transform

$$M : f(x) \to \tilde{f}(\alpha) = \int_0^\infty x^{\alpha-1} f(x) \, dx$$

$$M^{-1} : f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-\alpha} \tilde{f}(\alpha) \, d\alpha$$

$\boxed{\alpha \text{ is complex}}$

5 Hilbert Transform

$$H : f(t) \longrightarrow \hat{f}_H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt ,$$

$$H^{-1} : f(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{f}_H(x)}{x-t}\, dx .$$

6 Stieltjes Transform

$|\arg z| < \pi$

$$S : f(t) \longrightarrow \tilde{f}(z) = \int_0^{\infty} \frac{f(t)}{t+z}\, dt$$

Remark: The operation can be reversed but we do not get some simple integral formula as before so we do not describe this here.

7 Generalized Stieltjes Transform

$$S : f(t) \longrightarrow \tilde{f}_g(z) = \int_0^{\infty} \frac{f(t)}{(t+z)^g}\, dt$$

The same Remark as above conserning the reversed operation holds.

8 Legendre Transform

$$L : f(x) \longrightarrow \{\tilde{f}(n)\} , \quad \tilde{f}(n) = \int_{-1}^{1} P_n(x) f(x)\, dx$$

$P_n(x)$ is the Legendre polynomial of degree n!

$$L^{-1} : f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{(2n+1)}{2} \tilde{f}(n)}_{\text{Fourier coefficients}} P_n(x)$$

$\boxed{9}$ Jacobi Transform

$$\mathcal{J}: f(x) \longrightarrow \{f^{\alpha,\beta}(n)\},\ f^{\alpha,\beta}(n) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{\alpha,\beta}(x) f(x)\,d$$

$$\mathcal{J}^{-1}: f(x) = \sum_{n=0}^{\infty} \underbrace{(\delta_n)^{-1} f^{\alpha,\beta}(n)}_{\text{Fourier coefficients}} P_n^{\alpha,\beta}(x)$$

$$\left(\delta_n = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(n+\alpha+\beta+1)} \right)$$

Remark: Gegenbauer Transform is the specialcase $\alpha = \beta = \nu - \frac{1}{2}$

$\boxed{10}$ Laguerre Transform

$$L: f(x) \longrightarrow \{\tilde{f}_\alpha(n)\},\ \tilde{f}_\alpha(n) = \int_0^{\infty} e^{-x} x^{\alpha} L_n^{\alpha}(x) f(x)\,d$$

$L_n^{\alpha}(x)$ is the Laguerre Polynomial of degree n (≥ 0) and order α (>-1).

$$L^{-1}: f(x) = \sum_{n=0}^{\infty} \underbrace{(\delta_n)^{-1} \tilde{f}_\alpha(n)}_{\text{Fourier Coefficients}} L_n^{\alpha}(x)\,dx$$

$$\left(\delta_n = \binom{n+\alpha}{n} \Gamma(\alpha+1) \right)$$

$\boxed{11}$ Hermite Transform

$$H^*: f(x) \longrightarrow \{f_H(n)\},\ f_H(n) = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) f(x)\,d$$

$H_n(x)$ is the Hermite polynomial of degree

$$(H^*)^{-1}: f(x) = \sum_{n=0}^{\infty} \underbrace{\delta_n^{-1} f_H(n)}_{\text{Fourier coefficients}} H_n(x)$$

<u>Remark</u>: In <u>my</u> opinion all the transforms 8-11 are just special cases of our previous theory for generalized Fourier series but the author Debnath seems not to discuss in such terms