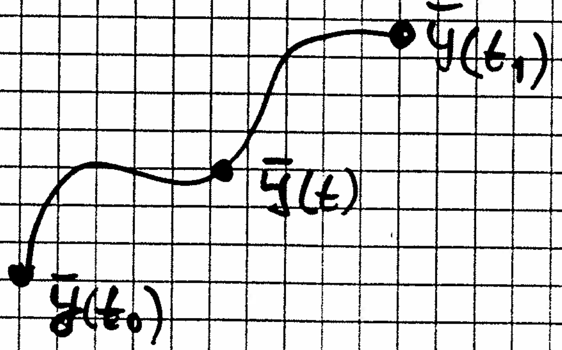
LECTURE 4

1. The Lagrangian

Consider a mechanical system



$\bar{y} = \bar{y}(t)$, i.e.,

$$\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \\ \vdots \\ y_n = y_n(t) \end{cases}$$

$y_1, y_2, \dots, y_n$ are called <u>generalized coordinates</u>

$\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n$ are called <u>generalized velocities</u>.

Moreover, we consider the generalized <u>kinetic energy</u>

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(y_1, \dots, y_n)\, \dot{y}_i \dot{y}_j$$

and the (generalized) <u>potential energy</u>

$$V = V(t, y_1, y_2, \dots, y_n, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_n)$$

Then we define the Lagrangian as

$$L = L(t, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n) = T - V$$

2. Hamilton's principle.

HP The motion of a mechanical system from time t_0 to t_1 is such that the functional

$$\mathcal{J} = \mathcal{J}(y_1, y_2, \dots, y_n) = \int_{t_0}^{t_1} L(t, y_1, \dots y_n, \dot{y}_1, \dots, \dot{y}_n)\,dt$$

is stationary for the functions $y_1(t), \dots, y_n(t)$ which describe the time evolution of the system.

Remark 1: In mathematical terms HP is often stated as

$$\delta \int_{t_0}^{t_1} L\,dt = 0.$$

$\int_{t_0}^{t_1} L\,dt$ is called the action integral.

Remark 2 In other terms Hamilton's principle means that among all possible paths from $\overline{y}(t_0)$ to $\overline{y}(t_1)$ the actual motion take place along the path that affords an extreme value to the integral $\int_{t_0}^{t_1} L\,dt$.

Remark 3 By using calculus of variations we find that $y_i(t)$ must satisfy Euler equation

$$(*) \qquad L_{y_i} - \frac{d}{dt}\left(L_{\dot{y}_i}\right) = 0, \quad i = 1, 2, \dots, n.$$

In this connection (*) are called Lagrange's equations.

According to Lagrange's equations (Remark 3) we have

$$(*) \quad L_{y_i} - \frac{d}{dt}(L_{\dot{y}_i}) = 0, \; i = 1, 2, \ldots, n.$$

Assume now that L is independent of t i.e. $L_t = 0$. Then, according to $(*)$, we have

$$\frac{d}{dt}\left(L - \sum_1^n \dot{y}_i L_{\dot{y}_i}\right) = 0, \text{ i.e.,}$$

$$(**) \quad -L + \sum_1^n \dot{y}_i L_{\dot{y}_i} = C.$$

$(**)$ is called a conservation law and the quantity

$$\boxed{H = -L + \sum_1^n \dot{y}_i L_{\dot{y}_i}}$$

is called the Hamiltonian of the system. It usually represent the total energy of the system.

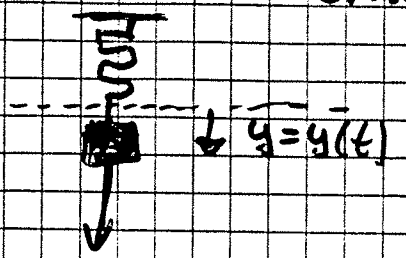
The above investigation means that If L is independent of time, then the total energy is conserved.

Remark: $\frac{d}{dt}\left(L - \sum_1^n \dot{y}_i L_{\dot{y}_i}\right) = L_t + \underline{\sum_1^n L_{y_i} \cdot \dot{y}_i} + \underline{\sum_1^n L_{\dot{y}_i} \frac{d}{dt}(\dot{y}_i)}$

$- \underline{\sum_1^n \dot{y}_i \frac{d}{dt} L_{\dot{y}_i}} - \underline{\sum_1^n \frac{d}{dt}(\dot{y}_i) L_{\dot{y}_i}} = 0$

4. Some examples

EX 1 (Harmonic Oscillator) STRING CONSTANT k



MASS m

HERE WE HAVE

$$T = \frac{1}{2} m \dot{y}^2 \qquad (\text{KINETIC ENERGY})$$

and

$$V = \frac{1}{2} k y^2 . \qquad (\text{POTENTIAL ENERGY})$$

Therefore

$$L = T - V = \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \; ($$

and

$$\mathcal{J}(y) = \int_{t_0}^{t_1} \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \right) dt .$$

$\delta \mathcal{J} = 0$ exactly when Lagrange's equation

$$L_y - \frac{d}{dt}(L_{\dot{y}}) = -ky - \frac{d}{dt}(m\dot{y}) = 0$$

is satisfied, i.e., when

$$(*) \quad m \ddot{y}(t) + k y(t) = 0 \qquad (\text{Newton's second law})$$

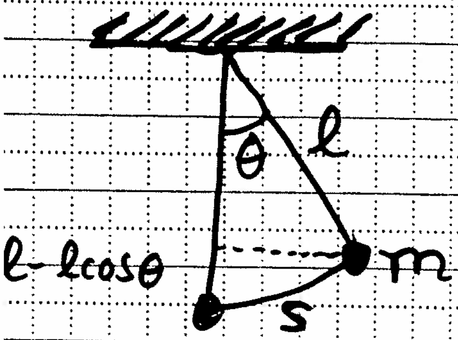
Hamilton's principle says that the actual motion is governed by $(*)$ with the solution

$$y(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

Finally we have

$$H = -L + \dot{y} L_{\dot{y}} = \frac{1}{2} k y^2 + \frac{1}{2} m \dot{y}^2$$

EX2 (Simple Pendlum)



Here we have

$$T = \frac{1}{2}m(\dot{s})^2 = \frac{1}{2}ml^2\dot{\theta}^2,$$

$$V = mg(l - l\cos\theta).$$

According to Hamilton's principle the motion takes place so that $\delta J = 0$, where

$$J(\theta) = \int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1}\left(\frac{1}{2}ml^2\dot{\theta}^2 - mg(l - l\cos\theta)\right)dt.$$

Moreover $\delta J = 0$ if the Lagrange's equation

$$L_\theta - \frac{d}{dt}L_{\dot{\theta}} = -mgl\sin\theta - ml^2\ddot{\theta} = 0$$

is satisfied, i.e., if

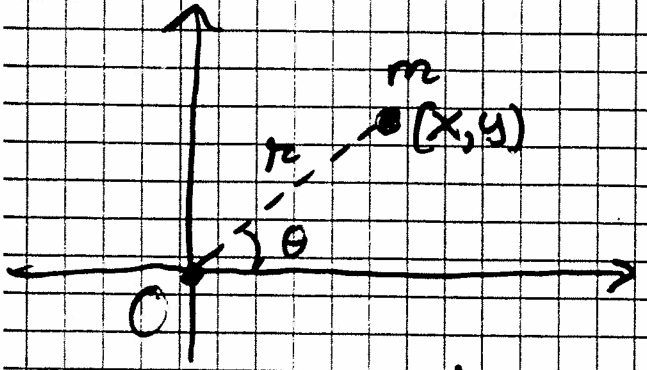
$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}.$$

This is the actual governing equation of the system.

In this case we have

$$H = -L + \dot{\theta}L_{\dot{\theta}} = mg(l - l\cos\theta) + \frac{ml^2\dot{\theta}^2}{2}.$$

EX 3 (Motion in a Central Force Field).



$$F = -\frac{k}{r^2}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\boxed{\begin{matrix} r = r(t) \\ \theta = \theta(t) \end{matrix}}$$

Here we have

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\left(\frac{d}{dt}(r\cos\theta)\right)^2 + \left(\frac{d}{dt}(r\sin\theta)\right)^2\right)$$

$$= \frac{1}{2}m(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2 + \frac{1}{2}m(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2 =$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and

$$V = -\frac{k}{r}.$$

According to Hamilton's principle we have $\delta J = 0$, where

$$J(r,\theta) = \int_{t_0}^{t_1} \overbrace{\left(\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}\right)}^{L} dt,$$

and this holds if the Lagrange's equations

$$L_\theta - \frac{d}{dt}(L_{\dot{\theta}}) = -\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

and

$$L_r - \frac{d}{dt}L_{\dot{r}} = mr\dot{\theta}^2 - \frac{k}{r^2} - \frac{d}{dt}(m\dot{r}) = 0$$

, i.e.,

$$\begin{cases} mr^2\dot{\theta} = C_0 \\ m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0 \end{cases}$$

This system can be solved exactly see Ex. 5.13

5. The canonical formalism (case $n=1$)

Consider the action integral

$$\mathcal{J}(y) = \int_{t_0}^{t_1} L(t, y, \dot{y})\, dt$$

and its corresponding Lagrange's equation

$$L_y - \frac{d}{dt} L_{\dot{y}} = 0 \qquad (1)$$

We define a new variable p, called the canonical momentum, by

$$\rightarrow \quad p = L_{\dot{y}}(t, y, \dot{y}). \qquad (2)$$

Solve this equation for $\dot{y}$ (Possible if e.g. $L_{\dot{y}\dot{y}} \neq 0$!) and we get

$$\dot{y} = \phi(t, y, p)$$

In particular, the Hamiltonian H can be written

$$H = -L + \dot{y} L_{\dot{y}} = -L(t, y, \phi(t,y,p)) + \phi(t,y,p)\,p$$

EX 4 Consider a particle of mass m moving in 1 dimension with potential energy $V(y)$. Then

$$L = \frac{1}{2} m \dot{y}^2 - V(y).$$

Hence $\quad p = L_{\dot{y}} = m\dot{y} \quad$ (usual momentum!)

$$H = -L + \phi \cdot p = -\frac{1}{2} m \left(\frac{p}{m}\right)^2 + V(y) + \frac{p}{m} \cdot p \quad \text{i.e.}$$

$$H = \frac{1}{2} \frac{p^2}{m} + V(y)$$

(= Total energy written in terms of position and momentum)

We also note that

$$\frac{\partial H}{\partial p} = -\underbrace{\frac{\partial L}{\partial \dot{y}}}_{p}\frac{\partial \phi}{\partial p} + \phi + p\frac{\partial \phi}{\partial p} = \dot{y}$$

$$\frac{\partial H}{\partial y} = -\frac{\partial L}{\partial y} - \underbrace{\frac{\partial L}{\partial \dot{y}}}_{p}\frac{\partial \phi}{\partial y} + p\frac{\partial \phi}{\partial y} = -L_y \overset{(1)}{=} -\frac{d}{dt}\underbrace{L_{\dot{y}}}_{p} = -\dot{p}$$

Thus Lagrange's equation (1) can be written as a system of equations

$$(*)\quad \begin{cases} \dot{y} = \dfrac{\partial H}{\partial p}(t, y, p), \\ \dot{p} = -\dfrac{\partial H}{\partial y}(t, y, p). \end{cases}$$

__DEFINITION__ $(*)$ is called __Hamilton's__ equations.

__EX 5__ Consider the Harmonic Oscillator from Example 1. Then

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 \quad (\text{Lagrangian})$$

$$p = L_{\dot{y}} = m\dot{y} \quad \text{i.e.} \quad (\text{momentum})$$

$$\dot{y} = \frac{p}{m} = \phi(t, y, p)$$

$$H = -L + \dot{y}L_{\dot{y}} = \quad (\text{Hamiltonian})$$

$$-\frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{1}{2}ky^2 + \frac{p}{m}p = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}ky^2$$

Hamilton's equations in this case are

$$(*)\quad \begin{cases} \dot{y} = \dfrac{\partial H}{\partial p}(t,y,p) = \dfrac{p}{m}, \\ \dot{p} = -\dfrac{\partial H}{\partial y}(t,y,p) = -ky. \end{cases}$$

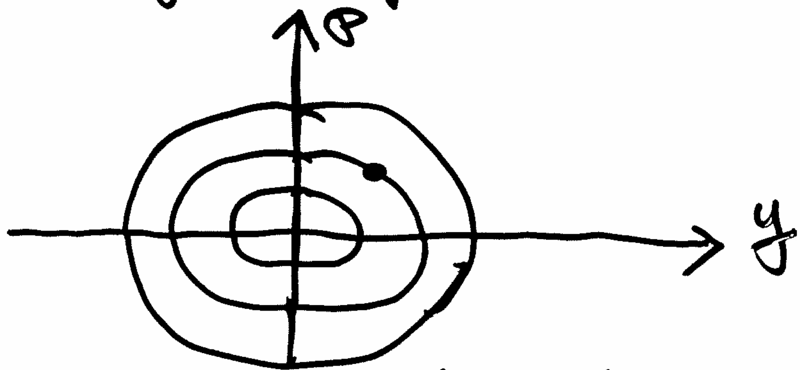
We solve this system in the yp-plane = the so called <u>phase plane</u>:
we divide and get

$$\frac{dp}{dy} = \frac{-ky}{p/m} \Leftrightarrow p\,dp + km\,y\,dy = 0$$

Thus

$$p^2 + kmy^2 = c \quad, c \text{ constant}$$

Geometrically this means a family of ellipses in the phase plane



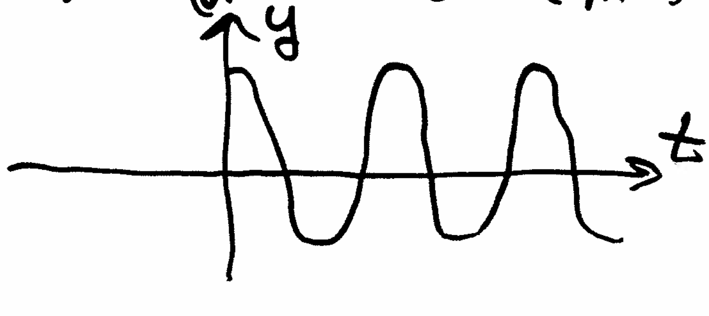
This is the paths the system evolves along in position-momentum spaces. also called the <u>phase space</u>.

<u>Remark</u> The Euler equation corresponding to $(*)$ is

$$\ddot{y} + \frac{k}{m}y = 0$$

with solutions $y(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

so the solutions can also be represented in the ty-plane as follows:



6. The canonical formalism (the general case)

For the general action integral

$$\mathcal{J}(y_1, \ldots, y_n) = \int_{t_0}^{t_1} L(t, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\, dt$$

we define the n canonical momenta

$$p_i = L_{\dot{y}_i}(t, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n), \quad i = 1, 2, \ldots, n$$

Solve this system of equation for $\dot{y}_1, \ldots, \dot{y}_n$ to obtain that

$$\dot{y}_i = \phi_i(t, y_1, \ldots, y_n, p_1, \ldots p_n), \quad i = 1, 2, \ldots n$$

The Hamiltonian H is defined by

$$H = -L + \sum_1^n \dot{y}_i L_{\dot{y}_i} = -L(t, y_1, \ldots, y_n, \phi_1, \ldots, \phi_n) + \sum_{i=1}^n \phi_i(t, y_1, \ldots, y_n, p_1, \ldots p_n) p_i$$

<u>Hamilton's equations</u>:

$$(*) \quad \begin{cases} \dot{y}_i = \dfrac{\partial H}{\partial p_i}(t, y_1, \ldots, y_n, p_1, \ldots, p_n) \\ \dot{p}_i = -\dfrac{\partial H}{\partial y_i}(t, y_1, \ldots, y_n, p_1, \ldots, p_n) \quad i = 1, 2, \ldots, n \end{cases}$$

$(*)$ is a system of $2n$ first order ODE with the unknown functions $y_1, \ldots, y_n$ and the corresponding unknown momentums $p_1, \ldots, p_n$.

$(*)$ carries the same information as the corresponding n Euler (Lagrange) equations.

$$(**) \quad L_{y_i} - \frac{d}{dt} L_{\dot{y}_i} = 0, \quad i = 1, 2, \ldots, n.$$

7. Lagrange Multiplier Rule (in $\mathbb{R}^n$)

Theorem: Let f and g be differentiable functions with $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ not both zero. If (x_0, y_0) provides an extreme value to f subject to the constraint $g(x,y) = 0$, then there exists a constant λ such that

$$\begin{cases} F'_x = 0, \\ F'_y = 0, \\ F'_\lambda = 0, \end{cases}$$

where $F(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$.

Remark A similar theorem holds for a general function $f = f(x_1, x_2, \dots, x_n)$ under m constraints.

EX: $n = 3$, $m = 2$. Write the constraints as $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$ and consider

$$F(x,y,z,\lambda_1,\lambda_2) = f(x,y,z) + \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z)$$

A possible exteme value can be obtained by solving the system

$$\begin{cases} F'_x = 0 \\ F'_y = 0 \\ F'_z = 0 \\ F'_{\lambda_1} = 0 \\ F'_{\lambda_2} = 0 \end{cases}.$$

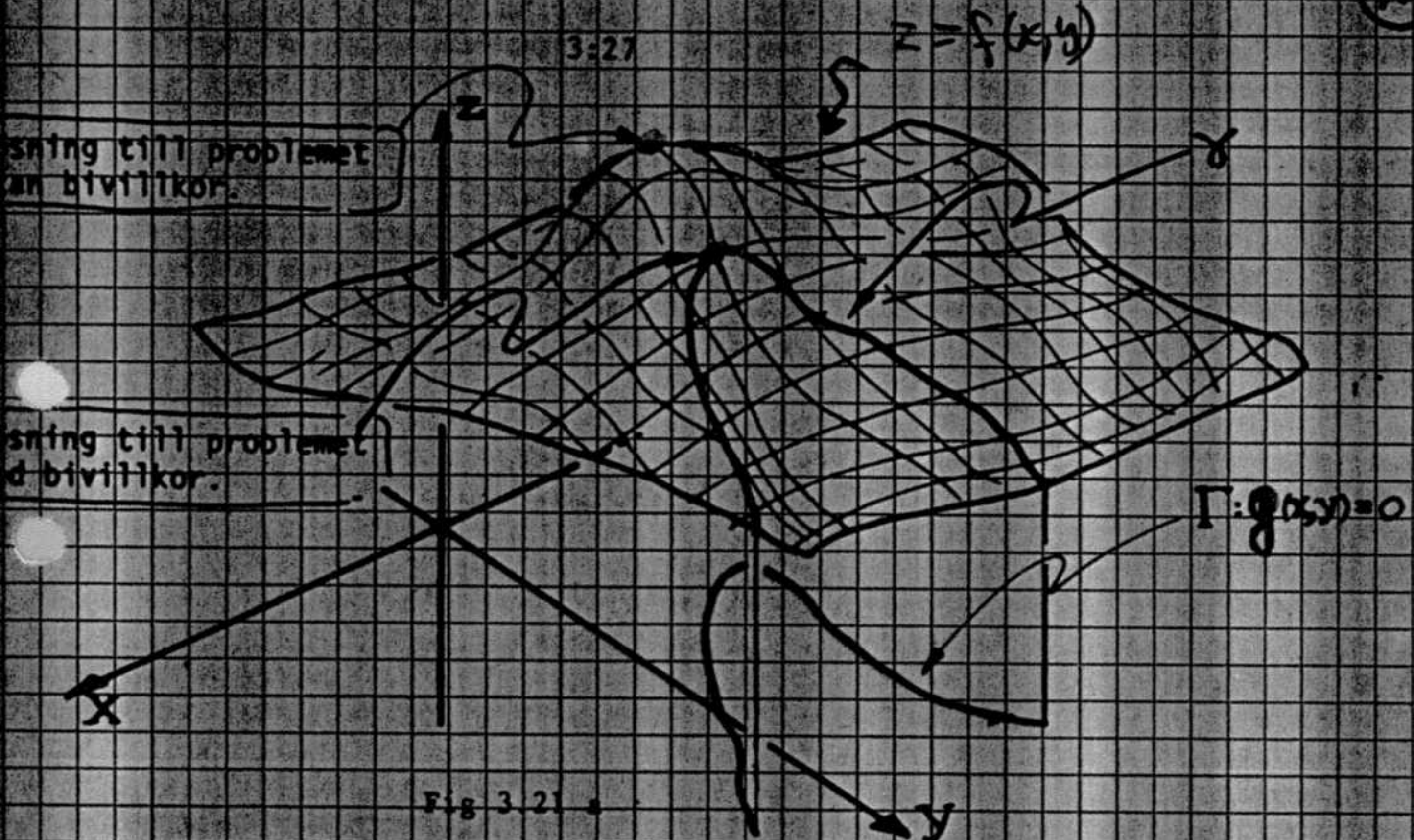
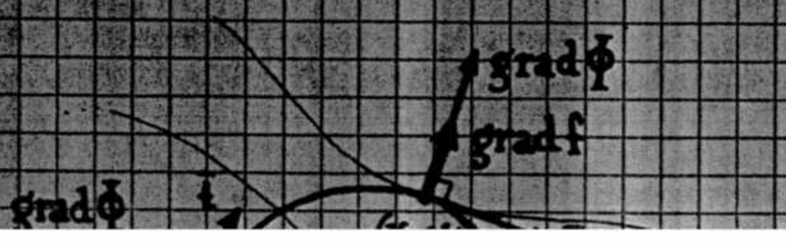
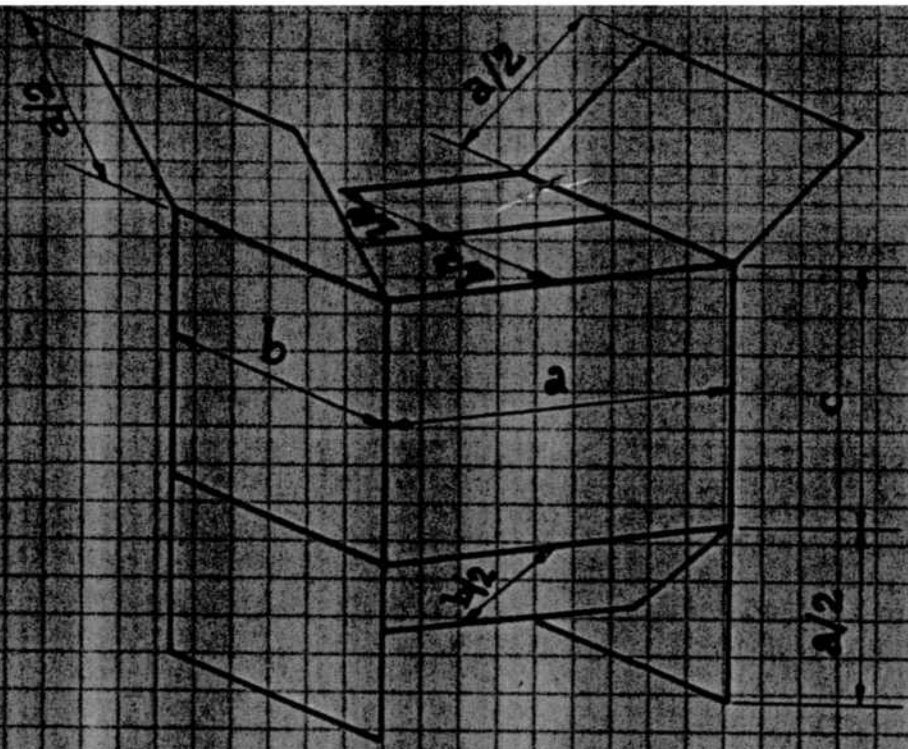


Fig 3.21 a

I många fall är det svårt eller praktiskt ogenomförbart att lösa ut någon variabel ur bivillkoret. Med den metod som vi nu skall gå igenom och som kallas Lagranges multiplikatormetod (efter den franske matematikern Joseph Lagrange 1736 - 1813) så får vi ett elegant och effektivt verktyg att behandla dessa problem.

Geometriskt representeras åter bivillkoret $\phi(x,y) = 0$ av en kurva Γ i xy-planet. Vi vill bestämma maximum av f när vi rör oss längs kurvan. Då vi rör oss längs denna anges f:s växande av riktningsderivatan $f'_{\bar{t}}$ där $\bar{t}$ är en normerad tangentvektor till kurvan $\phi(x,y) = 0$. Om (x_0,y_0) ger maximum av f så måste $f'_{\bar{t}}(x_0,y_0) = 0$. Men $f'_{\bar{t}} = \bar{t} \cdot \text{grad } f$. Alltså är $\bar{t}$ ortogonal mot grad f eller grad $f = \bar{0}$. Men $\bar{t}$ är alltid ortogonal mot grad ϕ.





Problem: We want to construct the box with the shape in the figure above with a fixed volyme V_0 and so that we need to use as little material as possible.

Solution. The problem means that we want to minimize the use of material

$$f(a,b,c) = 2ac + 2bc + 4ab$$

under the constraint

$$abc = V_0 \Leftrightarrow abc - V_0 = 0$$

According to LMR we consider

$$F(a,b,c,\lambda) = 2ac + 2bc + 4ab + \lambda(abc - V_0)$$

and solve the system

$$\begin{cases} F'_a = 2c + 4b + \lambda bc = 0 & (1) \\ F'_b = 2c + 4a + \lambda ac = 0 & (2) \\ F'_c = 2a + 2b + \lambda ab = 0 & (3) \\ F'_\lambda = abc - V_0 = 0 & (4) \end{cases}$$

Multiply (1), (2) and (3) by a, b and c respectively and we get

$$\begin{cases} 2ac + 4ab + \lambda V_0 = 0 & (5) \\ 2bc + 4ab + \lambda V_0 = 0 & (6) \\ 2ac + 2bc + \lambda V_0 = 0 & (7) \end{cases}$$

(5) $-$ (6) $\Rightarrow ac = bc \Rightarrow a = b \quad (c \neq 0)$

Insert $a = b$ into (6) and (7) and we get

$$\begin{cases} 2ac + 4a^2 + \lambda V_0 = 0 & (8) \\ 4ac \qquad\quad + \lambda V_0 = 0 & (9) \end{cases}$$

(8) $-$ (9) $\Rightarrow 4a^2 = 2ac \Rightarrow c = 2a$

Thus we choose $\begin{cases} b = a \\ c = 2a \end{cases}$

Moreover, according to (4),

$$a \cdot a \cdot 2a = V_0 \Leftrightarrow a = \left(\frac{V_0}{2}\right)^{1/3}$$

<u>**Answer:**</u> $a = b = \left(\frac{V_0}{2}\right)^{1/3}, \quad c = 2\left(\frac{V_0}{2}\right)^{1/3}.$

8. Isoperimetric problems

Problem: Minimize the functional

$$J(y) = \int_a^b L(x,y,y')\,dx$$

subject to the constraint

$$(1) \quad W(y) = \int_a^b G(x,y,y')\,dx = C$$

where $y \in C^2[a,b]$ and

$$(2) \quad y(a) = y_0, \ y(b) = y_1.$$

Method: (see [170]) We consider

$$J^*(y) = \int_a^b L(x,y,y')\,dx + \lambda \int_a^b G(x,y,y')\,dx$$

$$= \int_a^b (L + \lambda G)\,dx = \int_a^b L^*(x,y,y')\,dx$$

A necessary condition for minimum (extremum in general case) is that L^* satisfies Lagranges equation

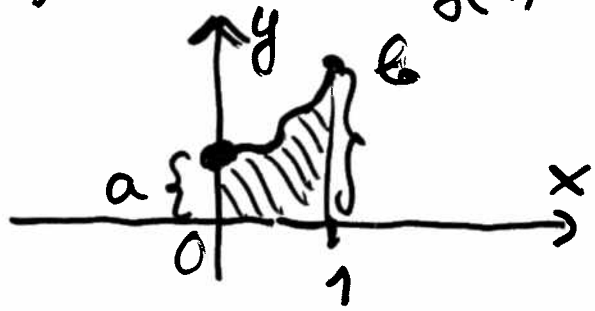
$$(3) \quad L^*_y(x,y,y') - \frac{d}{dx} L^*_{y'}(x,y,y') = 0$$

Remark: The problem and the corresponding method to solve it can be generalized to more dimensions and more constraints in the natural way.

Remark Usually the solution we get from (3) depends on two unknown constants and the multiplier λ. These can be determined by using (1) and (2).

9. Some examples

EX 7 Determine the shortest curve $y = y(x)$ which has area A below it and such that $y(0) = a$ and $y(1) = b$.



Solution: The problem is to minimize the functional

$$\mathcal{J}(y) = \int_0^1 \sqrt{1+(y'(x))^2}\,dx$$

subject to the constraint

$$W(y) = \int_0^1 y(x)\,dx = A$$

where

$$y(0) = a \text{ and } y(1) = b.$$

According to the "multiplier method" we consider

$$\mathcal{J}^*(y) = \int_0^1 \underbrace{\left(\sqrt{1+(y')^2} + \lambda y\right)}_{L^*(y)} dx$$

A necessary condition for minimum is that L^* satisfies Euler's equation

$$L^*_y - \frac{d}{dx}\left(L^*_{y'}\right) = \lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

This means that

$$\frac{y'}{\sqrt{1+(y')^2}} = \lambda x + c \Rightarrow y' = \frac{\lambda x + c}{\sqrt{1-(\lambda x + c)^2}}.$$

We integrate and find that

$$y = -\frac{1}{\lambda}\sqrt{1-(\lambda x + c)^2} + d \quad \text{i.e.}$$

$$(y\lambda - d_0)^2 + (\lambda x + c)^2 = 1 \qquad \underline{\text{a circle}}!$$

The unknown constants λ, d_0, c are determined from the (three) conditions

$$y(0) = a, \; y(1) = b \text{ and } \int_0^1 y(x)\,dx = A$$

Ex 8 See ex 6.1 in the book (shape of a hanging rope)

Ex 9 The famous <u>Schrödinger equation</u>

(se ex 6.2) $$-\frac{k^2}{2m}\nabla^2\psi + V\psi = -\lambda\psi$$

can (via Euler's equation) be written as the isoperimetric problem to

Extremize

$$J(\psi) = \iiint_D \left(\frac{k^2}{2m}(\psi_x^2 + \psi_y^2 + \psi_z^2) + V\psi^2\right)dx\,dy\,dz$$

subject to the constraint

$$\iiint_D \psi^2\,dx\,dy\,dz = 1$$

(Here $$L^* = \frac{k^2}{2m}(\psi_x^2 + \psi_y^2 + \psi_z^2) + V\psi^2 + \lambda\psi^2$$

Euler's equation

$$L^*_\psi - \frac{\partial}{\partial x}L^*_{\psi_x} - \frac{\partial}{\partial y}L^*_{\psi_y} - \frac{\partial}{\partial z}L^*_{\psi_z} = 0 \;)$$

Problems – Lecture 7

1. Consider the functional

$$J(y) = \int_a^b \left[r(t)\dot{y}^2 + q(t)y^2 \right] dt.$$

Find the Hamiltonian $H(t, y, p)$ and write down the canonical equations for the problem.

2.* Consider a system of n particles where m_i is the mass of the i:th particle and (x_i, y_i, z_i) is its position in the Euclidean three-space. The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2).$$

Assume that the system has a potential energy

$$V = V(t, x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)$$

such that the force acting on the i:th particle has components

$$F_i = -\frac{\partial V}{\partial x_i}, \quad G_i = -\frac{\partial V}{\partial y_i}, \quad H_i = -\frac{\partial V}{\partial z_i}.$$

Show that Hamilton's principle applied to this system yields the equations

$$\begin{cases} m_i \ddot{x}_i = F_i, \\ m_i \ddot{y}_i = G_i, \\ m_i \ddot{z}_i = H_i, \end{cases}$$

which are Newton's equations for a system of n particles.

3.* The American Post Office prescribe that the sum of the length and the (rectangular) circumference of the cross-section of a parcel is not allowed to exceed 100 inch. Determine the propotions of the parcel, within the permitted limits, which has the largest possible volume.
(Answer: 2:1:1)

4. Calculate the least distance from a point on the ellipse $x^2+4y^2=4$ to the line $x+y=4$.
(Answer: $(4-\sqrt{5})/\sqrt{2}$)

5.* Determine the equation of the shortest arc in the first quadrant that passes through (0,0) and (1,0) and encloses a prescribed area A with the x-axis, where $0<A\leq\pi/8$.

6. Find the extremals of the isoperimetr problem:

$$J(y)=\int_0^{\pi}(y')^2dx,\ y(0)=y(\pi)=0,$$

subject to the constraint

$$\int_0^{\pi}y^2dx=1.$$