Lecture 8 - Integral equations

1. Introductory Examples

Ex 1:

(a) $y(x) = x - \int_0^x (x-\xi) y(\xi) d\xi$,

(b) $y(x) = f(x) + \lambda \int_0^x k(x-\xi)\, y(\xi) d\xi$

($f(x)$ and $k(x)$ are fixed functions),

(c) $y(x) = \lambda \int_0^1 k(x,t) y(t) dt$, where

$$k(x,t) = \begin{cases} x(1-t), & x \le t \le 1, \\ t(1-x), & 0 \le t \le 1. \end{cases}$$

(d) $y(x) = \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi$,

(e) $y(x) = f(x) + \lambda \int_0^1 (1-3x\xi) y(\xi) d\xi$.

Ex 2: "Fredholm's equation".

$$\int_a^b k(x,\xi) y(\xi) d\xi + a(x) y(x) = f(x).$$

First Kind $\Leftrightarrow$ $a(x) \equiv 0$

Second Kind $\Leftrightarrow$ $a(x) \not\equiv 0$

$\boxed{\begin{array}{l} k(x,\xi) = \text{kernel} \\ f(x) = \text{outer force term} \\ y(x) = \text{"outsignal"} \end{array}}$

(Ivar Fredholm 1866-1927)

Ex3: "Volterra's equation".

$$\int_a^x k(x,\xi)\,y(\xi)\,d\xi + a(x)\,y(x) = f(x).$$

First Kind $\Leftrightarrow$ $a(x) \equiv 0$

Second Kind $\Leftrightarrow$ $a(x) \not\equiv 0$

(Vito Volterra 1860 - 1940)

Ex4: "Salesman control problem".

a = amount of "goods" at time $t=0$.

$k(t)$ = remaining percentage of goods after time t.

$u(t)$ = rate to buy new goods (goods/time)

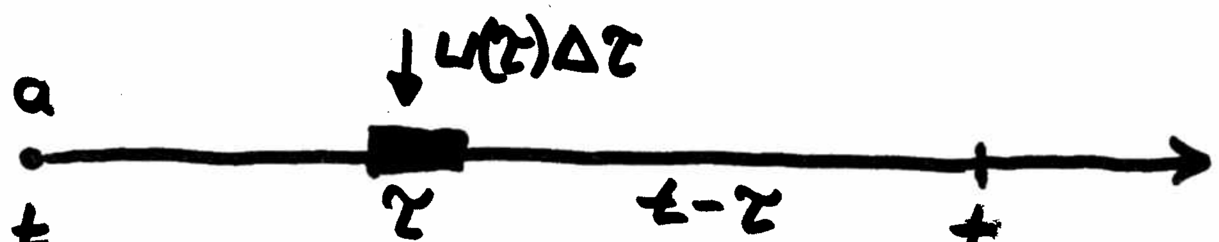
$u(\tau)\Delta\tau$ = amount of good bought in time $\Delta\tau$.

Total amount of "goods" in the shop at time t

$$a\,k(t) + \int_a^t k(t-\tau)\,u\,\tau\,d\tau$$

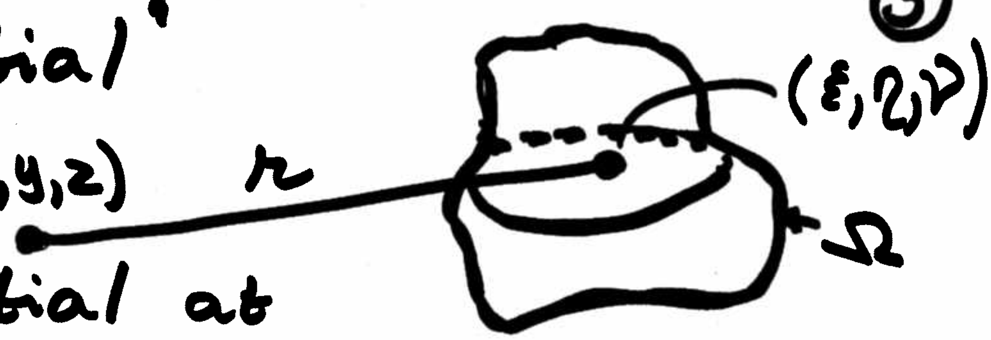
The salesman can have a constant amount of goods if

$$(*)\quad a\,k(t) + \int_0^t k(t-\tau)\,u(\tau)\,d\tau = c_0.$$



Ex5: "Potential"

$V(x,y,z)$



is the potential at the point (x,y,z) due to a massdistribution $\rho(\xi,\eta,\nu)$ in Ω. Then

- $$V(x,y,z) = -G \iiint_{\Omega} \frac{\rho(\xi,\eta,\nu)}{r}\, d\xi\, d\eta\, d\nu .$$

The inverse problem is to determine ρ from a given potential V. This is described by the integral equation

$$\nabla^2 V = 4\pi G \rho ,$$

i.e. Poisson's equation.

2. <u>Integral equations of convolution type</u>:

$$\underbrace{y(x) = f(x) + \int_0^x k(x-t)\, y(t)\, dt}_{k * y(x)} \qquad (*)$$

$k*y(x)$ = <u>convolution</u> between $k(x)$ and $y(x)$

Main technique to solve $(*)$:

Use Laplace transforms!

EX6: Solve

$$(*) \quad y(x) = x - \int_0^x (x-t)\, y(t)\, dt.$$

Sol: This equation is of convolution type with $f(x) = x$ and $k(x) = x$. We note that $\mathcal{L}(x) = \frac{1}{s^2}$ and Laplace transformation of $(*)$ gives

$$\mathcal{L}(y) = \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}(y), \text{ i.e.,}$$

$$\mathcal{L}(y) = \frac{1}{1+s^2}.$$

Thus $\underline{\underline{y(x)}} = \mathcal{L}^{-1}\left(\frac{1}{1+s^2}\right) = \underline{\underline{\sin x}}.$

EX7: Solve

$$y(x) = f(x) + \lambda \int_0^x k(x-t)\, y(t)\, dt$$

($f(x)$ and $k(x)$ are fixed functions).

Sol: Laplace transformation gives that

$$\mathcal{L}(y) = \mathcal{L}(f) + \lambda\, \mathcal{L}(k)\, \mathcal{L}(y), \text{ i.e.,}$$

$$\mathcal{L}(y) = \frac{\mathcal{L}(f)}{1 - \lambda \mathcal{L}(k)} \quad \text{so that}$$

$$\underline{\underline{y(x)}} = \underline{\underline{\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{1-\lambda \mathcal{L}(k)}\right)}}.$$

3. Relationship between Differential and Integral equations (first order)

Ex 8: Consider the ODE

(1) $y' = f(x,y)$, $y(x_0) = y_0$.

By integrating from x_0 to x we find that

(2) $\int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt$, i.e.,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.$$

On the other hand, if the IE (2) holds, then we see that $y(x_0) = y_0$ and, by differentiating,

$$y'(x) = f(x, y(x)),$$

which means that (1) holds!

■ In fact, it is possible to reformulate many initial and boundary problems as integral equations and vice versa. In general,

Initial problems, dynamical systems $\longleftrightarrow$ Volterra Eq.

Boundary value problems $\longleftrightarrow$ Fredholm Eq.

4. Picard's method

Problem: Solve

(*) $\quad y' = f(x,y)\,,\quad y(x_0) = A.$

Write (*) as an integral equation

$$y(x) = A + \int_{x_0}^{x} f(t, y(t))\,dt.$$

Choose an intial approximation $\boxed{y(x) = y_0(x)}$ and calculate

$$y_1(x) = A + \int_{x_0}^{x} f(t, y_0(t))\,dt,$$

$$y_2(x) = A + \int_{x_0}^{x} f(t, y_1(t))\,dt,$$

$$\vdots$$

$$y_n(x) = A + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

$$\vdots$$

$y(x) \approx y_n(x)$ "Hopefully: $y(x) = \lim\limits_{n\to\infty} y_n(x)$"

Ex 9: Solve

$$y' = 2x(1+y),\quad y(0) = 0.$$

Sol: (via Picard's method)

$$y(x) = \int_0^x 2t(1+y(t))\,dt$$

choose $y_0 = 0$. Then

$$y_1(x) = \int_0^x 2t(1+0)\,dt = x^2,$$

$$y_2(x) = \int_0^x 2t(1+t^2)\,dt = t^2 + \frac{t^4}{4},$$

$$y_3(x) = \int_0^x 2t\left(1+t^2+\frac{t^4}{4}\right)dt = x^2+\frac{x^4}{2}+\frac{x^6}{2\cdot 3},$$

$$\vdots$$

$$y_n(x) = x^2+\frac{x^4}{2}+\frac{x^6}{3!}+\cdots\cdots+\frac{x^{2n}}{n!}.$$

$$\lim_{n\to\infty} y_n(x) = e^{x^2}-1$$

Remark: Note that $y(x)=e^{x^2}-1$ is the exact solution of [redacted]. (Prove that!)

5. Two Lemmas

Lemma 1: $\int_a^x \int_a^s f(y)\,dy\,ds = \int_a^x f(y)(x-y)\,dy,$

whenever f is continuous for $x \geq a$.

Proof: Let $F(s)=\int_a^s f(y)\,dy$. Then, by using integration by parts, we have

$$\int_a^x \int_a^s f(y)\,dy\,ds = \int_a^x F(s)\,ds = \int_a^x 1\cdot F(s)\,ds =$$

$$[s\,F(s)]_a^x - \int_a^x s\,F'(s)\,ds = xF(x) - aF(a)$$

$$-\int_a^x s\,F'(s)\,ds = x\int_a^x f(y)\,dy - \int_a^x s\,f(s)\,ds =$$

$$\int_a^x x\,f(y)\,dy - \int_a^x y\,f(y)\,dy = \int_a^x (x-y)\,f(y)\,dy.$$

Lemma 2: "Leibniz formula".

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} u(x,t)\,dx\right) = \int_{a(t)}^{b(t)} u'_t(x,t)\,dx +$$

$$+\, u(b(t),t)\,b'(t) - u(a(t),t)\,a'(t).$$

Proof": Consider $G(t,a,b) = \int_a^b u(x,t)\,dt$, where

$$\begin{cases} a = a(t) \\ b = b(t) \end{cases}.$$

The chain rule gives

$$\frac{d}{dt} G(t,a,b) = G'_t(t,a,b) + G'_a(t,a,b)\,a'(t) +$$

$$+\, G'_b(t,a,b)\,b'(t) = \int_a^b u'_t(x,t)\,dt$$

$$-\, u(a(t),t)\,a'(t) + u(b(t),t)\,b'(t).$$

Ex10: $F(t) = \int_{\sqrt{t}}^{t^2} \sin xt\,dx$. Then

$$F'(t) = \int_{\sqrt{t}}^{t^2} x\cos xt\,dt + \sin t^3 \cdot 2t -$$

$$-\, \sin t^{3/2} \cdot \frac{1}{2\sqrt{t}}.$$

6. Relationship between Differential equations and Integral equations (second order)

Ex 11: $\begin{cases} u'' + q(x)u = f(x) \\ u(a) = u_0, \quad u'(a) = u_1. \end{cases}$ $(*)$

we integrate and find that

$$u'(x) - u_1 = \int_a^x \big(f(y) - q(y)u(y)\big)\,dy.$$

Integrate once more!

$$\int_a^x u'(s)\,ds = \int_a^x u_1\,ds + \int_a^x \int_a^s \big(f(y) - q(y)u(y)\big)\,dy.$$

Thus, by Lemma 1,

$$u(x) - u_0 = u_1(x-a) + \int_a^x \big(f(y) - q(y)u(y)\big)(x-y)\,dy,$$

which can be rewritten as

$$u(x) = \underbrace{u_0 + u_1(x-a) + \int_a^x f(y)(x-y)\,dy}_{F(x)} + \int_a^x \underbrace{q(y)(x-y)}_{k(x,y)}u(y)\,dy$$

This means that $(*)$ in fact can be rewritten as the Volterra equation

$$u(x) = F(x) + \int_a^x k(x,y)\,u(y)\,dy.$$

Remark: Ex 11 shows how an intial value problem (i.e. a differential equation equipped with intial values) can be transformed to a (Volterra type) integral equation. In the next example we will show that an integral equation can be transformed to a differential equation.

EX 12: (*) $y(x) = \lambda \int_0^1 k(x,t)\, y(t)\, dt$, where

$$k(x,t) = \begin{cases} x(1-t), & x \le t \le 1, \\ t(1-x), & 0 \le t \le x. \end{cases}$$

$$\because\ y(x) = \lambda \int_0^x t(1-x)\, y(t)\, dt + \lambda \int_x^1 x(1-t)\, y(t)\, dt$$

Hence, by Lemma 2,

$$y'(x) = \lambda \int_0^x -t\, y(t)\, dt + \lambda x(1-x)\, y(x)$$
$$+ \lambda \int_x^1 (1-t)\, y(t)\, dt - \lambda x(1-x)\, y(x).$$

By using Lemma 2 once more we find that

$$y''(x) = -\lambda x\, y(x) - \lambda(1-x)\, y(x) = -\lambda\, y(x).$$

Moreover, we see that $y(0) = y(1) = 0$.
Thus, (*) can equivalently be rewritten as the boundary value problem

$$\begin{cases} y''(x) + \lambda\, y(x) = 0 \\ y(0) = y(1) = 0. \end{cases}$$

(Remark: This boundary value we remembe was very important in the Fourier theory in "the method of separation of variables")

7. A general technique to solve Fredholm's integral equation (with separable kernel).

(1) $y(x) = f(x) + \lambda \int_a^b k(x,\xi)\, y(\xi)\, d\xi$,

where

$$k(x,\xi) = \sum_1^n \alpha_n(x)\beta_n(\xi).$$ $\boxed{\text{Separable kernel!}}$

Then

(2) $y(x) = f(x) + \lambda \sum_1^n \underbrace{\int_a^b \beta_n(\xi)\, y(\xi)\, d\xi}_{c_n}\ \alpha_n(x)$

Remark: (2) is a solution of (2) if we know the numbers c_n, so the question is:

- How to find the numbers c_n?

▪ We multiply by $\beta_j(x)$, integrate and find that

(3) $\underbrace{\int_a^b y(x)\beta_j(x)\, dx}_{c_j} = \underbrace{\int_a^b f(x)\beta_j(x)\, dx}_{f_j} + \lambda \sum_1^n c_n \underbrace{\int_a^b \alpha_n(x)\beta_j(x)\, dx}_{a_{jn}}$

$(j = 1, 2, 3, \ldots, n)$.

- This is just a linear system of n equations and n unknown!

In matrix form this system can be written as

$$I\bar{c} = \bar{f} + \lambda A\bar{c}, \text{ i.e.,}$$

$$(I - \lambda A)\bar{c} = \bar{f},$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \bar{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \text{ and } \bar{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

- Some well-known facts from the basis course in linear algebra:

$$B\bar{c} = \bar{f}$$

1. $\begin{cases} \bar{b} = \bar{0}, \det B \neq 0 \Rightarrow \bar{c} = \bar{0} \\ \bar{b} = \bar{0}, \det B = 0 \Rightarrow \infty \text{ many solutions} \end{cases}$

2. $\begin{cases} \bar{b} \neq \bar{0}, \det B \neq 0 \Rightarrow 1 \text{ solution} \\ \bar{b} \neq \bar{0}, \det B = 0 \Rightarrow 0 \text{ or } \infty \text{ many solutions.} \end{cases}$

- The famous <u>Fredholm's alternative theorem</u> is just a reformulation of these facts (with $B = I - \lambda A$)

<u>Ex 13</u>: Solve

$$(*) \quad y(x) = \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi.$$

Note that $1 - 3x\xi = 1 \cdot 1 + (-3x)\,\xi$

Here we can put

$$\begin{cases} \alpha_1(x) = 1, & \alpha_2(x) = -3x \\ \beta_1(\xi) = 1 & \beta_2(\xi) = \xi \end{cases}$$

we have

$$A = \begin{pmatrix} \int_0^1 \beta_1 \alpha_1 \, dx & \int_0^1 \beta_1 \alpha_2 \, dx \\ \int_0^1 \beta_2 \alpha_1 \, dx & \int_0^1 \beta_2 \alpha_2 \, dx \end{pmatrix} = \begin{pmatrix} \int_0^1 1 \cdot 1 \, dx & \int_0^1 1(-3x)\,dx \\ \int_0^1 x \cdot 1 \, dx & \int_0^1 x(-3x)\,dx \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3/2 \\ 1/2 & -1 \end{pmatrix}$$

$$\det(I - \lambda A) = \det\begin{pmatrix} 1-\lambda & 3/2\,\lambda \\ -\lambda/2 & 1+\lambda \end{pmatrix} = 1 - \frac{\lambda^2}{4} = 0$$

$\Leftrightarrow \lambda = \pm 2.$

Fredholm's alternative theorem now gives the following possibilities

- $\lambda \neq \pm 2$. (*) has only the trivial solution $y(x) \equiv 0$.
- $\lambda = 2$. Then the system $(I - \lambda A)\bar{c} = \bar{0}$ reads:

$$\begin{cases} -c_1 + 3c_2 = 0 \\ -c_1 + 3c_2 = 0 \end{cases}$$

with solutions $c_2 = A$, $c_1 = 3A$ (A is any constant). Therefore, according to (2),

$$y(x) = 0 + 2(3A \cdot 1 + A(-3x)) = 6A(1-x) = B(1-x)$$

∴ $\boxed{y(x) = B(1-x) \text{ are the solutions of } (*) \text{ when } \lambda = 2.}$

- $\lambda = -2$. Then the system $(I - \lambda A)\bar{c} = \bar{0}$ reads:

$$\begin{cases} 3c_1 - 3c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$$

with solutions $c_2 = A$ and $c_1 = A$.
Therefore, again by (2),

$$y(x) = 0 - 2(A \cdot 1 + A(-3x)) = B(1 - 3x)$$

$$\boxed{\text{$y(x) = B(1-3x)$ are the solutions of $(*)$ when $\lambda = -2$.}}$$

EX 14: Solve

$$(**) \quad y(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi.$$

Again we note that

$$\det(I - \lambda A) = 0 \Leftrightarrow \lambda \neq \pm 2.$$

Fredholm's alternative theorem now gives all the following possibilities:

1. Assume that $b_1 = \int_0^1 f(x) \cdot 1\, dx \neq 0$ or $b_2 = \int_0^1 f(x) \cdot x\, dx \neq 0$. $\lambda \neq \pm 2$

Then $(**)$ has the unique solution

$$y(x) = f(x) + \lambda(c_1 \cdot 1 + c_2(-3x)) = f(x) + \lambda(c_1 - 3c_2 x),$$

where c_1 and c_2 is the unique solution of the system

$$\begin{cases}(1-\lambda)c_1 + \frac{3}{2}\lambda c_2 = \int_0^1 f(x)dx \\ -\frac{1}{2}\lambda c_1 + (1+\lambda)c_2 = \int_0^1 x f(x)dx .\end{cases}$$

2. Assume that $\int_0^1 f(x)\cdot 1 dx \neq 0$ or $\int_0^1 f(x) x dx \neq 0$ and $\lambda = -2$.

We have to consider the system

$$\begin{cases}3c_1 - 3c_2 = \int_0^1 f(x)dx , \\ c_1 - c_2 = \int_0^1 x f(x)dx ,\end{cases}$$

which has

- no solution if $\int_0^1 f(x)dx \neq 3\int_0^1 x f(x)dx$.
- ∞ many solutions if $\int_0^1 f(x)dx = 3\int_0^1 x f(x)dx$.

Put $3c_2 = A \Rightarrow 3c_1 = A + \int_0^1 f(x)dx$.

$\therefore$ According to (2) we see that the solutions of $(\cancel{*}**)$ in this case are

$$y(x) = f(x) - 2\left[c_1 1 + c_2(-3x)\right] =$$

$$f(x) - 2\left[\left(\frac{A}{3} + \frac{1}{3}\int_0^1 f(x)dx\right) + \frac{A}{3}(-3x)\right] =$$

$$f(x) - \frac{2}{3}\int_0^1 f(x)dx + A\left(2x - \frac{2}{3}\right).$$

3. Assume that $\int_0^1 f(x)dx \neq 0$ or $\int_0^1 f(x) x dx \neq 0$ and $\lambda = 2$.

We have to consider the system

$$\begin{cases} -c_1+3c_2=\int_0^1 f(x)\,dx, \\ -c_1+3c_2=\int_0^1 x f(x)\,dx, \end{cases}$$

which has

- no solution if $\int_0^1 f(x)\,dx \neq \int_0^1 x f(x)\,dx$.
- ∞ many solutions if
$$\int_0^1 f(x)\,dx = \int_0^1 x f(x)\,dx.$$

Put $c_2 = A \Longrightarrow c_1 = 3A - \int_0^1 f(x)\,dx$.

$\therefore$ According to (2i) we find that the solutions of (**) in this case are

$$y(x) = f(x) + 2\left[c_1\cdot 1 + c_2(-3x)\right] =$$
$$f(x) + 2\left[\left(3A - \int_0^1 f(x)\,dx\right) + A(-3x)\right] =$$
$$f(x) - 2\int_0^1 f(x)\,dx + \underbrace{6A}_{B}(1-x).$$

4. $\int_0^1 f(x)\cdot 1\,dx = \int_0^1 f(x)\, x\,dx = 0$ and $\lambda \neq \pm 2$.

(Then $c_1 = c_2 = 0$ is the unique solution).

$\therefore$ $y(x) = f(x)$ is the unique solution of (**).

5. $\int_0^1 f(x)\cdot 1\,dx = \int_0^1 f(x)x\,dx = 0$ and $\lambda = -2$ (17)

$$\begin{cases} 3c_1 - 3c_2 = 0 \\ c_1 - c_2 = 0 \end{cases} \Leftrightarrow c_1 = c_2 = A.$$

(**) has ∞ many solutions namely

$$y(x) = f(x) - 2(A\cdot 1 + A(-3x)) = f(x) + B(1-3x).$$

6. $\int_0^1 f(x)\cdot 1\,dx = \int_0^1 f(x)\cdot x\,dx = 0$ and $\lambda = 2$.

$$\begin{cases} -c_1 + 3c_2 = 0 \\ -c_1 + 3c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_2 = A \\ c_1 = 3A \end{cases}.$$

(**) has ∞ many solutions, namely

$$y(x) = f(x) + 2(3A\cdot 1 + A(-3x)) = f(x) + B(1-x).$$

8. Integralequations with a symmetric kernel.

We consider

$$(*) \quad y(x) = \lambda \int_a^b k(x,\xi)\, y(\xi)\, d\xi,$$

where the kernel $k(x,\xi)$ is symmetric, i.e.

$$k(x,\xi) = k(\xi, x).$$

The following important theorem yields:

Theorem 1: Consider (*) with symmetric kernel. Then

(i) λ_n eigenvalue with eigenvector y_n
λ_m — " — — " — y_m

$$\lambda_n \neq \lambda_m \Rightarrow \int_a^b y_m(x)\, y_n(x) = 0. \quad \boxed{y_m \perp y_n}$$

(ii) The eigenvalues are real.

(iii) If the kernel is not separable, then there are infinitely many eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$

$$0 < |\lambda_1| \leq |\lambda_2| \leq \dots\dots$$

$$\lim_{n\to\infty} |\lambda_n| = \infty.$$

(iv) To each eigenvalue there corresponds at most finitely many independent eigenfunctions.

(**Remark** Because we know about the close connection to differential equations this is not so surprising!)

Ex 15: Consider

$$(*) \qquad y(x) = \lambda \int_0^1 k(x,\xi)\, y(\xi)\, d\xi,$$

where

$$k(x,\xi) = \begin{cases} x(1-\xi), & x \le \xi \le 1, \\ \xi(1-x), & 0 \le \xi < x. \end{cases}$$

According to Ex 12 $(*)$ is equivalent to

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y(0) = y(1) = 0 \end{cases} .$$

$\lambda > 0$ $\qquad y(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y(1) = 0 \Rightarrow \sin\sqrt{\lambda} = 0$$

$\therefore$ $\lambda = \lambda_n = n^2\pi^2$ (eigenvalues)

$y_n(x) = \sin n\pi x$ (eigenfunctions)

■ This means that $(*)$ has nontrivial solutions if and only if $\lambda = \lambda_n = n^2\pi^2$. The untrivial solutions are $a_n \sin n\pi x$.

Proof of (ii):

$$y_m(x) = \lambda_m \int_a^b k(x,\xi)\, y_m(\xi)\, d\xi \,,$$

$$y_n(x) = \lambda_n \int_a^b k(x,\xi)\, y_n(\xi)\, d\xi$$

$$\int_a^b y_m(x)\, y_n(x)\, dx = \lambda_m \int_a^b y_n(x) \int_a^b k(x,\xi)\, y_m(\xi)\, d\xi\, dx =$$

$$= \lambda_m \int_a^b \left(\int_a^b y_n(x)\, k(x,\xi)\, dx \right) y_m(\xi)\, d\xi = \begin{bmatrix} k(x,\xi) = \\ k(\xi, x) \end{bmatrix} =$$

$$= \lambda_m \int_a^b \left(\int_a^b k(\xi, x)\, y_n(x)\, dx \right) y_m(\xi)\, d\xi =$$

$$\lambda_m \int_a^b \frac{1}{\lambda_n}\, y_n(\xi)\, y_m(\xi)\, d\xi =$$

$$= \frac{\lambda_m}{\lambda_n} \int_a^b y_m(\xi)\, y_n(\xi)\, d\xi = \frac{\lambda_m}{\lambda_n} \int_a^b y_m(x)\, y_n(x)\, dx.$$

We conclude that

$$\underbrace{\left(1 - \frac{\lambda_m}{\lambda_n}\right)}_{\neq 0} \int_a^b y_m(x)\, y_n(x)\, dx = 0,$$

so that

$$\int_a^b y_m(x)\, y_n(x)\, dx = 0.$$

The proof is complete.

9. Hilbert-Schmidt theory for solving the (Fredholm) integral equation

(*) $$y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt$$

Hilbert-Schmidt's lemma: Assume that $g(t)$ is a continuous function on $[a,b]$ and put

$$F(x) = \int_a^b k(x,t)\, g(t)\, dt,$$

where $k(x,t)$ is symmetric. Then $F(x)$ can be expanded in Fourier series as

$$F(x) = \sum_1^{\infty} c_n y_n(x),$$

where $y_n(x)$ are the (normalized) eigenfunctions of

(**) $$y(x) = \lambda \int_a^b k(x,t)\, y(t)\, dt.$$

Hilbert-Schmidt's theorem: Let λ be different from all eigenvalues of (**) and let $y(x)$ be the solution of (*). Then

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} y_n(x),$$

where λ_n and y_n are the eigenvalues and eigenfunctions of (**) and

$$f_n = \int_a^b f(x)\, y_n(x)\, dx.$$

(= the Fourier coefficients).

EX 16: Solve

$$y(x) = x + \lambda \int_0^1 k(x,\xi)\, y(\xi)\, d\xi,$$

where $\lambda \neq n^2\pi^2$, $n = 1, 2, \dots$ and

$$k(x,\xi) = \begin{cases} x(1-\xi), & x \le \xi \le 1, \\ \xi(1-x), & 0 \le \xi < x. \end{cases}$$

Sol: According to Ex 15 we know that the normalized eigenfunctions are $y_n(x) = \sqrt{2}\sin n\pi x$ corresponding to the eigenvalues $\lambda_n = n^2\pi^2$.

Moreover,

$$f_n = \int_0^1 x\sqrt{2}\sin n\pi x\, dx = \frac{(-1)^{n+1}\sqrt{2}}{n\pi}.$$

$\therefore$ In view of the Hilbert-Schmidt theorem, the solutions are

$$y(x) = x + \frac{\sqrt{2}\,\lambda}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n(n^2\pi^2-\lambda)}\sin n\pi x,$$

$$\boxed{\lambda \neq n^2\pi^2}$$

Proof of Hilbert-Schmidt's theorem:

According to (*) we have

$$(**) \quad y(x) - f(x) = \lambda \int_a^b k(x,\xi)\, y(\xi)\, d\xi$$

Therefore, by the Hilbert-Schmidt lemma,

$$y(x) - f(x) = \sum_1^\infty c_n y_n(x),$$

where (the Fourier coefficients are)

■ $$c_n = \int_a^b \big(y(x) - f(x)\big) y_n(x)\, dx = \int_a^b y(x) y_n(x)\, dx - f_n.$$

By multiplying (**) by $y_n(x)$ and integrating we find that

$$\int_a^b f(x) y_n(x)\, dx = f_n + \lambda \int_a^b \Big(\int_a^b k(x,\xi) y(\xi)\, d\xi \Big) y_n(x)\, dx$$

$$= [\text{symmetry}] = f_n + \lambda \int_a^b \Big(\int_a^b k(\xi, x) y_n(x)\, dx \Big) y(\xi)\, d\xi$$

$$= f_n + \frac{\lambda}{\lambda_n} \int_a^b y_n(\xi) y(\xi)\, d\xi = f_n + \frac{\lambda}{\lambda_n} \int_a^b y_n(x) y(x)\, dx,$$

$$\therefore \int_a^b y(x) y_n(x)\, dx = \frac{f_n}{1 - \frac{\lambda}{\lambda_n}} = \frac{f_n \lambda_n}{\lambda_n - \lambda}$$

and we conclude from ■ that

$$c_n = \frac{f_n \lambda_n}{\lambda_n - \lambda} - f_n = \frac{\lambda f_n}{\lambda_n - \lambda}$$

$$\therefore y(x) = f(x) + \sum_1^\infty \frac{\lambda f_n}{\lambda_n - \lambda} y_n(x)$$

$$f(x) + \lambda \sum_1^\infty \frac{f_n}{\lambda_n - \lambda} y_n(x).$$

Probems - Lecture 8

1.* Use the Laplace transform to solve

(a) $y(x) = f(x) + \lambda \int_0^x e^{x-t} y(t)\,dt$,

(b) $x = \int_0^x e^{x-t} y(t)\,dt$.

2. Prove the Leibniz formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} F'_x(x,y)\,dy + F(x,b(x))\,b'(x) -$$

$$- F(x,b(x))\,a'(x)$$

for differentiating an integral with respect to a variable appearing both in the integrand and the limits of integration.

Hint: Call the integral $g(x,a,b)$, where $a=a(x)$, $b=b(x)$ and use the chain rule.

3. Reformulate the initial value problem

$$u'' - \lambda u = f(x),\ x>0,\ u(0)=1,\ u'(0)=0,$$

as a Volterra integral equation.

4. Reformulate the boundary value problem

$$u'' + \lambda u = 0,\ 0<x<\ell,\ u(0)=0,\ u(\ell)=0$$

as a Fredholm integral equation

Hint: Integrate from 0 to x twice.

5.* Solve the integral equations:

(a) $y(x) = x^2 + \int_0^1 (1-3xt)\, y(t)\, dt$,

(b) $y(x) = x^2 + \lambda \int_0^1 (1-3xt)\, y(t)\, dt$

for all values of λ where it is possible.

6.* (a) Reformulate the integral equation

(**) $\quad y(x) = \lambda \int_0^1 k(x,t)\, y(t)\, dt,$

where

$$k(x,t) = \begin{cases} x(1-t), & x \le t \le 1, \\ t(1-x), & 0 \le t < x, \end{cases}$$

as a boundary value problem.

(b) Find all eigenvalues and eigenfunctions of (**).

(c) Solve the integral equation

$$y(x) = x + \lambda \int_0^1 k(x,t)\, y(t)\, dt,$$

where $k(x,t)$ is defined in (a) and $\lambda \ne n^2\pi^2, n \in \mathbb{Z}_+$.