Lecture 9

Introduction to dynamical systems, chaos, Bifurcations, etc..

1. Introduction

A dynamical system is a phenomenon which changes with the time, e.g., the position of a pendlum, the weather, the amounts of predators ("rov-fiskar") and preys ("betesfiskar") in a lake, etc... The traditional way to describe a dynamical system is by using a linear system of differential equations. In this case we have a fairly simple theory to solve the problems (see our section 7. However, a more realistic modelling often implies that we must handle nonlinear systems of differential equations. Here it is much more complicated to describe the long-run behaviour but by using computers and existing theory we can sometimes obtain the long-run behavior as an <u>attractor</u> of the system. Many times we get <u>bifurcations</u> or <u>chaos</u>. Chaos means that it is difficult (or impossible) to judge the long-run behavior of the system; small changes in the initial data give dramatic changes in the long-run behavior. The attractors are often described as <u>fractals</u>, which are special "self-similar" sets (a small part has the same structure as the whole set). Such attractors are sometimes called <u>strange attractors</u>.

<u>Remark</u>: This fascinating and important area of mathematics is still under rapid development and we can wait a lot of fundamental discoveries from the research.

<u>Remark</u>: Two important concepts to study in this connection are stability and bifurcation (= the title of section 6).

2. Discrete dynamical systems.

Let $f: V \to V$ be a continuous function.

The orbit : $x_0, x_1, x_2, \ldots\ldots$

x_0 is a startvalue

$x_1 = f(x_0)$

$x_2 = f(x_1)$

$\vdots$

$x_{n+1} = f(x_n)$

$\vdots$

x_n = the state of the dynamical system in step n (time n).

A main problem: What can we say about x_n for big values of n?

We can obtain chaos if small changes in x_0 implies big (or "unpredictable") changes in x_n.

The attractor is the state of the dynamical system for big values of n.

Example 1: Verhulst's population model

$$f_r(x) = (1+r)x - rx^2 \quad (r > 0 \text{ is a parameter})$$

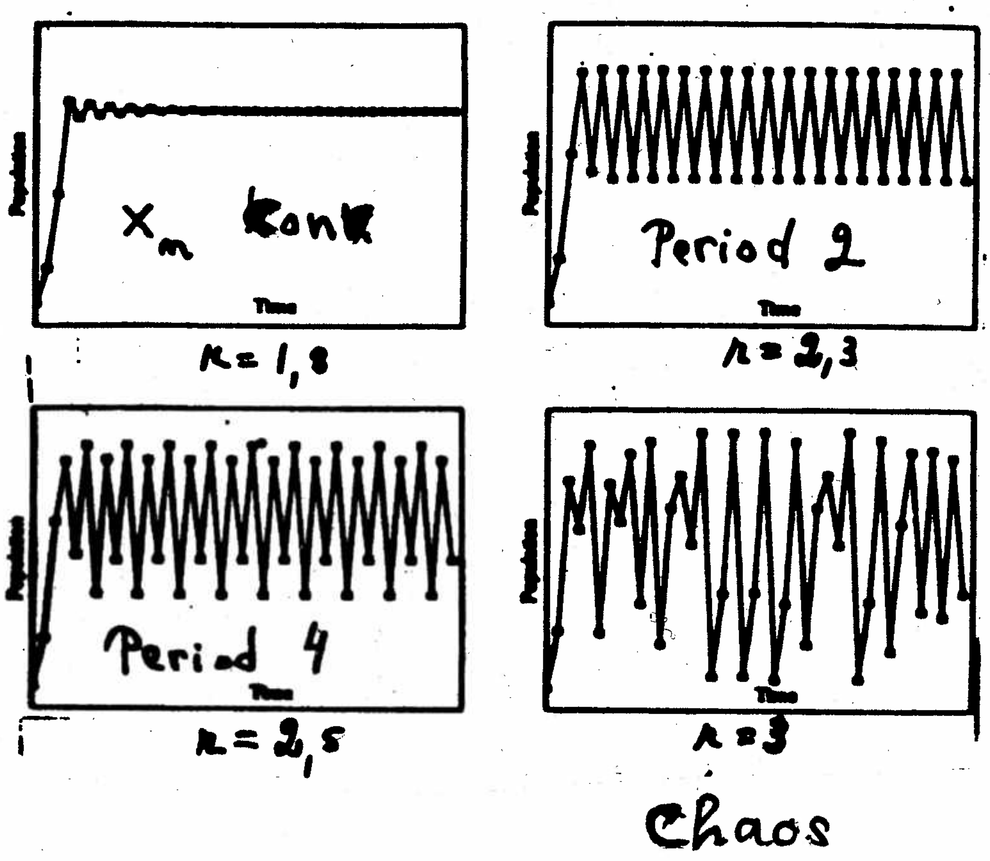
This is one useful description of the growth of a population.

The orbit $x_0, x_1, x_2, \ldots$ is obtained in the following way:
Consider a starting value x_0 at year 0.

$$x_{n+1} = f_r(x_n), \quad n = 0, 1, 2, \ldots$$

x_n = the population year n.

Different values of r gives very different values of the orbit (Try yourself with your calculator!)



The orbit converges to the attractor. Describe the attractor in the different cases!

It can be proved that the following holds:

There exists a sequence of parameters $r_1, r_2, r_3, \dots$ such that

* If $0 < r < r_1$, then $\{x_n\}$ converges to one value
* If $r_1 < r < r_2$, then $\{x_n\}$ converges to a periodic state with two values.
* If $r_2 < r < r_3$, then $\{x_n\}$ converges to a periodic state with four values.

⋮

* If $r_k < r < r_{k+1}$, then $\{x_n\}$ converges to a perodic state with 2^k values

$$r_1 < r_2 < r_3 < \dots \qquad < r_\infty = 2.570\dots$$

* For $r > r_\infty$ we have chaos. Then the long-run behaviour of $\{x_n\}$ is not periodic and the orbit is extremely depending on x_0. This is not the case when $r < r_\infty$.

Feigenbaum's constant:

$$\lim_{k \to \infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = 4,669201\dots$$

This formula holds for several other functions f_k!

3. On the Liapunov exponent λ.

* $x_n = f(x_{n-1}) = f(f(x_{n-2})) = \cdots = \underbrace{f(\cdots\cdots f}_{n \text{ times}}(x_0)) \triangleq f_n(x_0)$

* $f_n'(x_0) = f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot \cdots \cdot f'(x_0)$

* $\frac{1}{n} \log|f_n'(x_0)| = \frac{1}{n} \sum_{k=0}^{n-1} \log|f'(x_k)| \longrightarrow \lambda$ as $n \to \infty$

This convergence can be proved by using a special mathematical technique.

λ is called the Liapunov exponent of the dynamical system

* Assume now that we have two different starting values x_0 and y_0. Then, by using the mean value theorem,

$$|f_n(x_0) - f_n(y_0)| = |x_0 - y_0| \cdot |f_n'(\xi_0)| \approx$$
$$\approx |x_0 - y_0| (e^{\lambda})^n \text{ for big values of } n$$

* Conclusion:

a) If $\lambda < 0$, then $\lim\limits_{n\to\infty} f_n(x_0) = \lim\limits_{n\to\infty} f_n(y_0)$, i.e. the dynamical system is not sensible for the starting value

b) If $\lambda > 0$, then $|f_n(x_0) - f_n(y_0)|$ converges to infinity with exponential growth

4/ Another example; Julia and Mandelbrot sets

Let z be a complex number and consider $f_c(z) = z^2 + c$. $z_{n+1} = f_c(z_n)$

* The case $c = 0$.

$|z_0| < 1 \Rightarrow \lim_{n\to\infty} z_{n+1} = 0$

$|z_0| > 1 \Rightarrow \lim_{n\to\infty} z_{n+1} = \infty$ (the complex infinity)

$|z_0| = 1 \Rightarrow |z_n| = 1 \quad n = 1, 2, 3, \ldots$

* The case $c \neq 0$

Consider $J_c = \{z_0 : z_n$ stay within a fixed boundary in the plane$\}$

This kind of boundary is called a Julia set

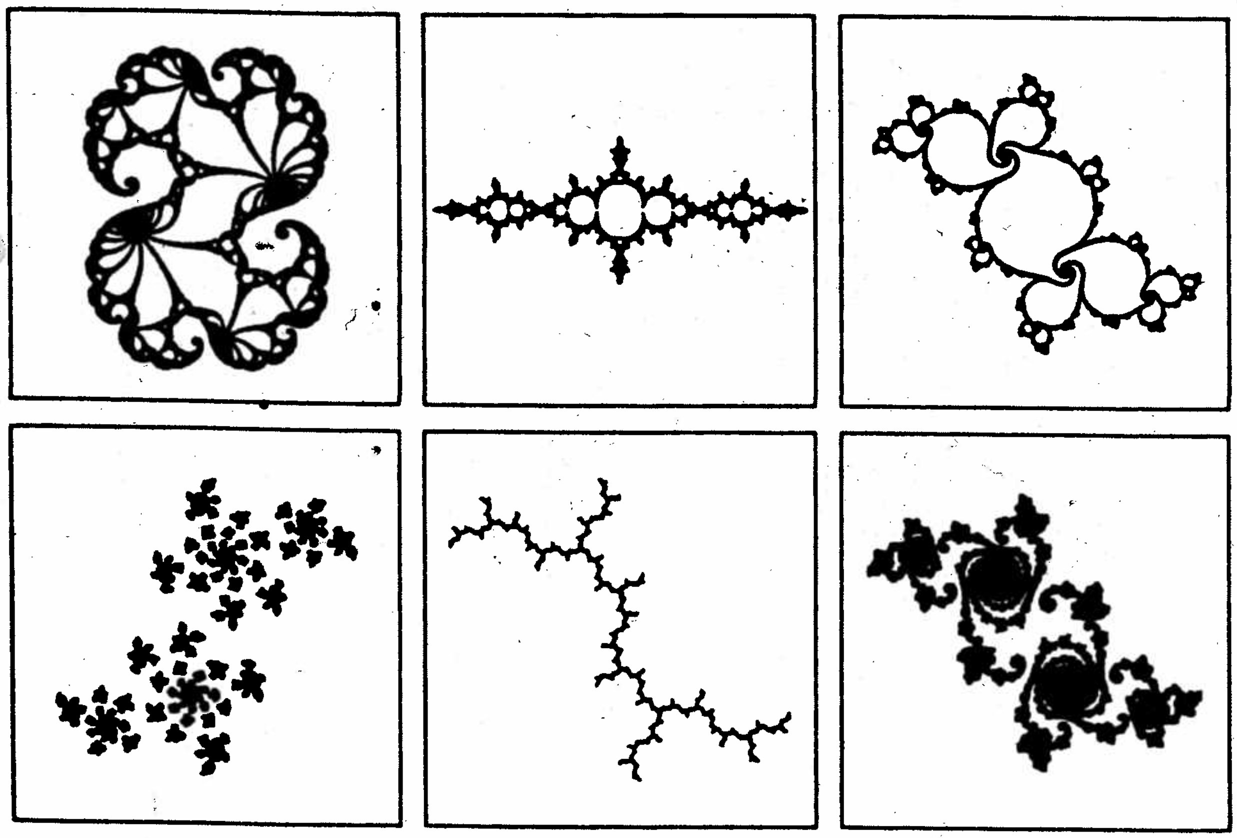


FIGURE [redacted] Julia sets, when iterated for various values of c in the expression $z^2 + c$ come in many forms, some connected and some fragmented.

= 6 pages from a "book" with infinite many pages

* In every case, depending on their starting points, some orbits stay bounded whereeas others streak off to infinity.
* It turns out that there are only two major types of Julia sets, no matter what c is. Either the area within a boundary is connected <u>or</u> is broken into infinite number of separate pieces to form a cloud of points of a fascinating fractal nature (selfsimilarity).
* <u>The Mandelbrot set</u> = the set of C such that the Julia sets are connected.

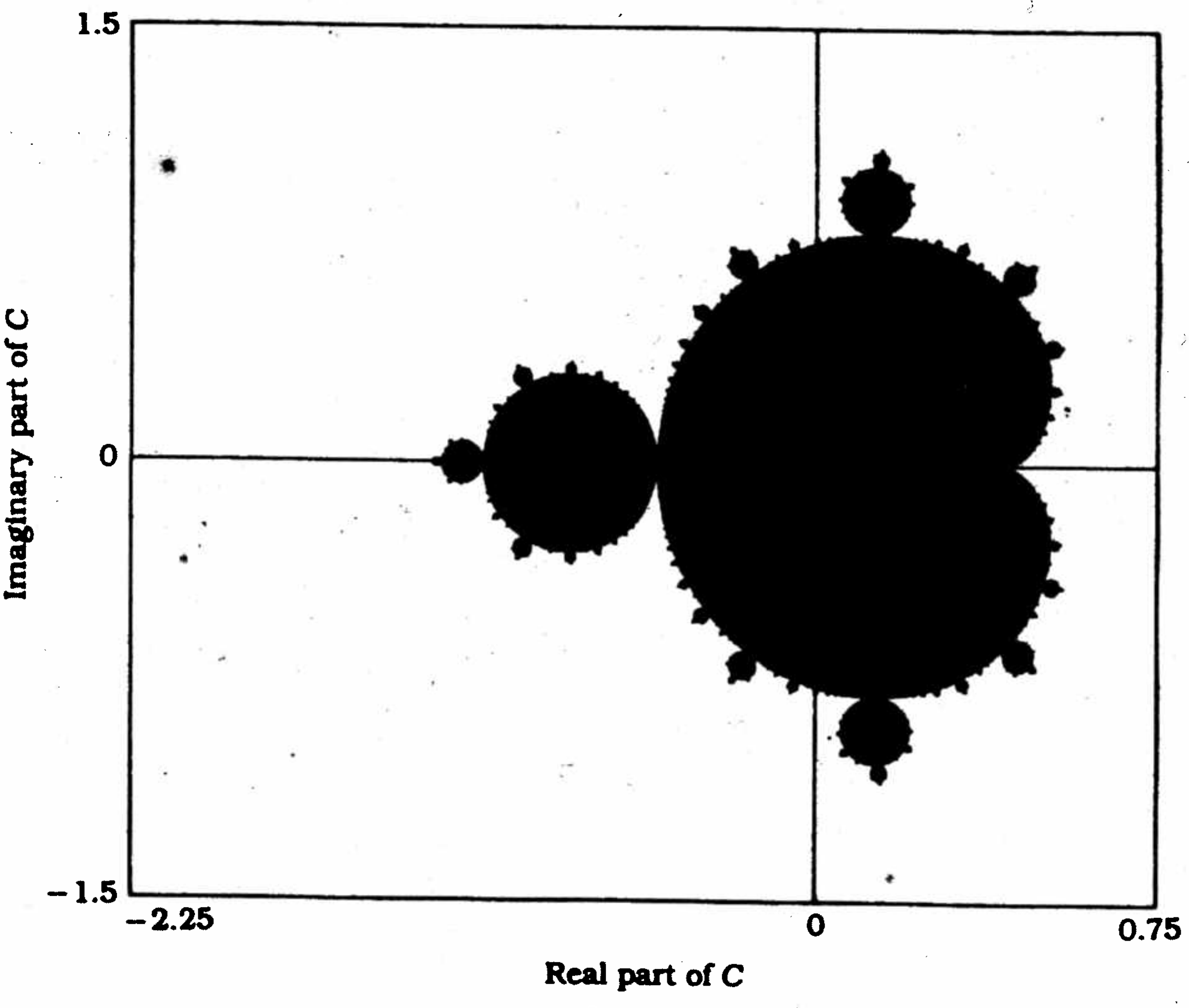


FIGURE [redacted] The Mandelbrot set (shown in black) extends from the cusp of the cardioid at $c = 0.25$ to the tip of its tail at $c = 2$.

The Mandelbrot set has detailed structure on all scales. Zooming in for a closer look shows fine structures that seem to get more and more fantastic and complicated as the magnification increases. Close-ups of its borders unveil a riot of tendrils and curlicues, yet everything is connected. A bewildering array of delicate filaments holds all the parts together *(see Figure 6.7)*.

Delving deeper and deeper also turns up miniature snowmen.

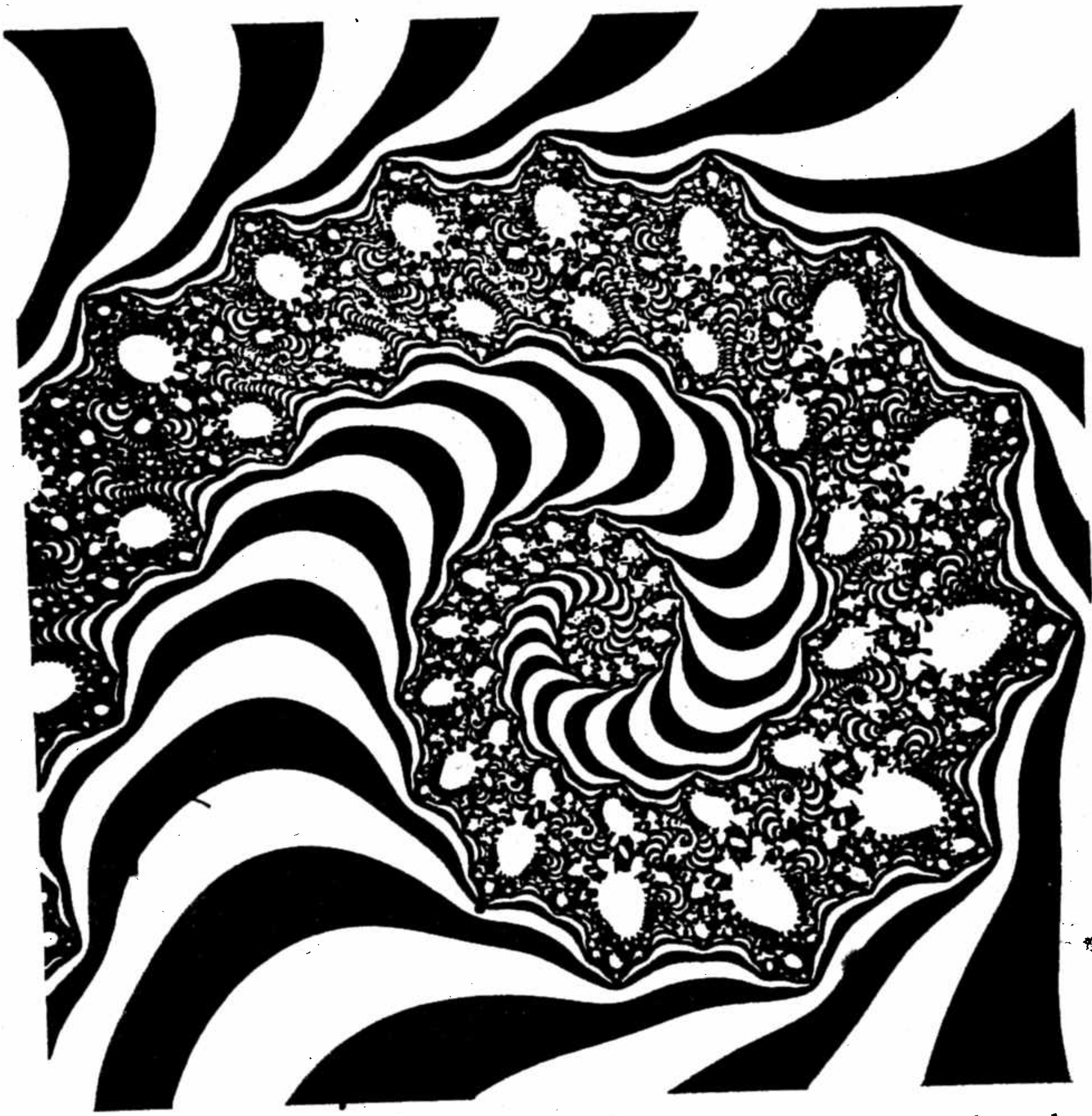


FIGURE [redacted] A blowup of a piece of the Mandelbrot set's boundary reveals a riot of filaments and small copies of the Mandelbrot set.

5 Continuous dynamical systems.

In the most general case we study a system of the type

$$\dot{\bar{x}} = F(\bar{x}),$$

where $\bar{x} = \bar{x}(t)$. For each fixed t, $x(t)$ is a vector in a finite or infinite dimensional vector space.

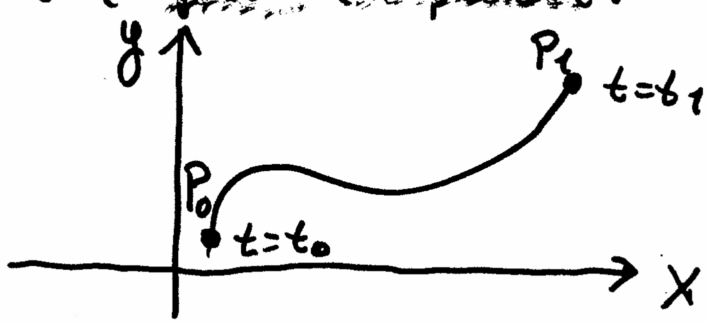
In this course we will mainly study systems of the form

$$(*) \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

where $x = x(t)$ and $y = y(t)$ are usual realvalued functions(Here we have $\bar{x} = (x,y)$ and $\bar{F} = (P,Q)$)

The solutions are defined on some interval $t_0 \leq t \leq t_1$ and they can be graphed in the state space.

Alternatively they can be represented as a parametric curve in the xy-plane also called the phase plane.



$P_0 = (x(t_0), y(t_0))$
$P_1 = (x(t_1), y(t_1))$.

The <u>equilibrium points</u> are obtained by solving the system

$$(**)\quad \begin{cases} P(x,y)=0, \\ G(x,y)=0. \end{cases}$$

Since $\dot{y}/\dot{x} = \frac{dy}{dx}$ we obtain (see (*))

$$(***)\quad \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

A <u>Phase portrait</u> of the system (*) is the totality of all the paths and equilibrium (= critical) points in the phase plane

<u>A quick way to obtain a phase portrait:</u>

1°. Find all equilibrium points by solving (**). Mark them in a xy plan!

2°. Let some standard program (e.g. REGSIM by Thomas Gustafsson) illustrate the solutions of (***).

In our section 11 we present a more careful way to create a phase portrait of our dynamical system (**)

In our next section we give several concrete examples of continuous dynamical systems and their phase portrait

6. Some introductory examples

Example 2: Consider the system

$$\begin{cases} \dot{x} = y, & (1) \\ \dot{y} = -x. & (2) \end{cases}$$

According to (1) it holds $\dot{y} = \ddot{x}$, which, inserted into (2) gives

$$\ddot{x} + x = 0$$

with the solutions

$$x(t) = a\cos t + b\sin t = a_0 \sin(t + \varphi_0).$$

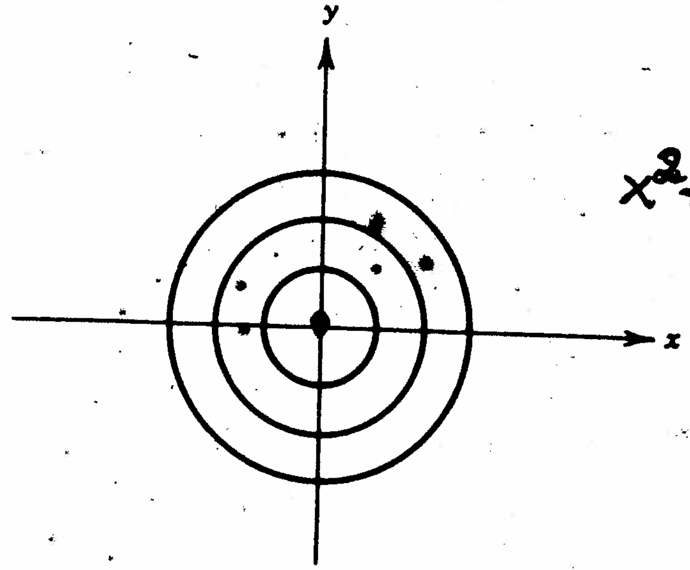
Moreover, by (1),

$$y(t) = -a\sin t + b\cos t = a_0 \sin(t + \varphi_1).$$

We also note that

$$x^2 + y^2 = (a\cos t + b\sin t)^2 + (-a\sin t + b\cos t)^2$$
$$= a^2(\cos^2 t + \sin^2 t) + b^2(\cos^2 t + \sin^2 t) = a^2 + b^2$$

Obviously the only equilibrium point is $(0,0)$.



$$x^2 + y^2 = c^2 \quad (c = \sqrt{a^2 + b^2})$$

[redacted] Phase portrait [redacted]

Remark: The relation $x^2+y^2=c^2$ can also be obtained in the following way:
According to (**) $\frac{dy}{dx}=-\frac{x}{y} \Leftrightarrow \int y\,dy=\int -x\,dx$

$$\Leftrightarrow \frac{y^2}{2}+\frac{x^2}{2}=c_0 \Leftrightarrow x^2+y^2=c^2 \ (\text{with } c=\sqrt{2c_0})$$

Example 3: Consider the system

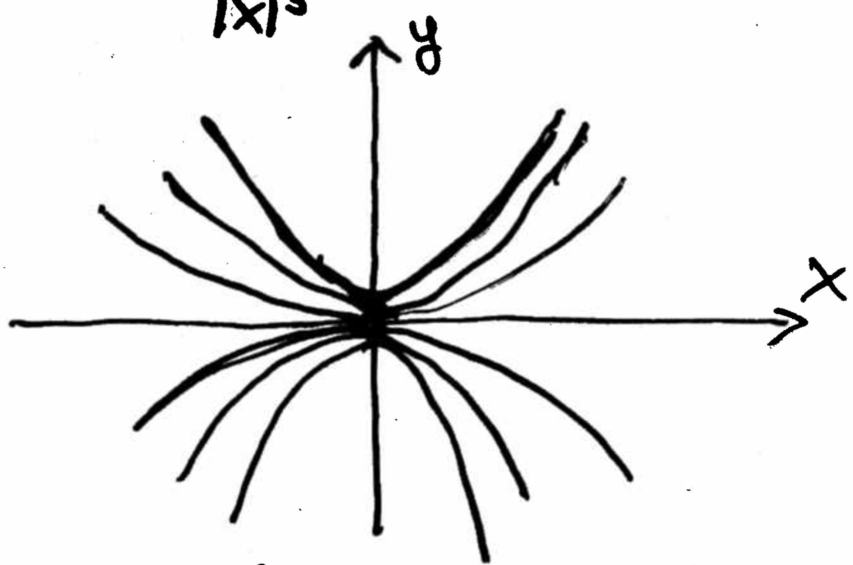
$$\begin{cases} \dot{x}=2x \\ \dot{y}=3y \end{cases}$$

This uncoupled system has the solution

$$x=c_1e^{2t},\ y=c_2e^{3t}$$

$$\frac{dy}{dx}=\frac{3y}{2x} \Leftrightarrow \int\frac{2dy}{y}=\int\frac{3dx}{x} \Leftrightarrow 2\ln|y|=3\ln|x|+c$$

$$\Leftrightarrow \ln\frac{y^2}{|x|^3}=c \Leftrightarrow y^2=c_0|x|^3.$$



Phase portrait

Example 4: Any second order differential equation

$$\ddot{x} = f(x, \dot{x}),$$

can be written as a system of first order equations (of the type (*)) namely as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = f(x, y). \end{cases}$$

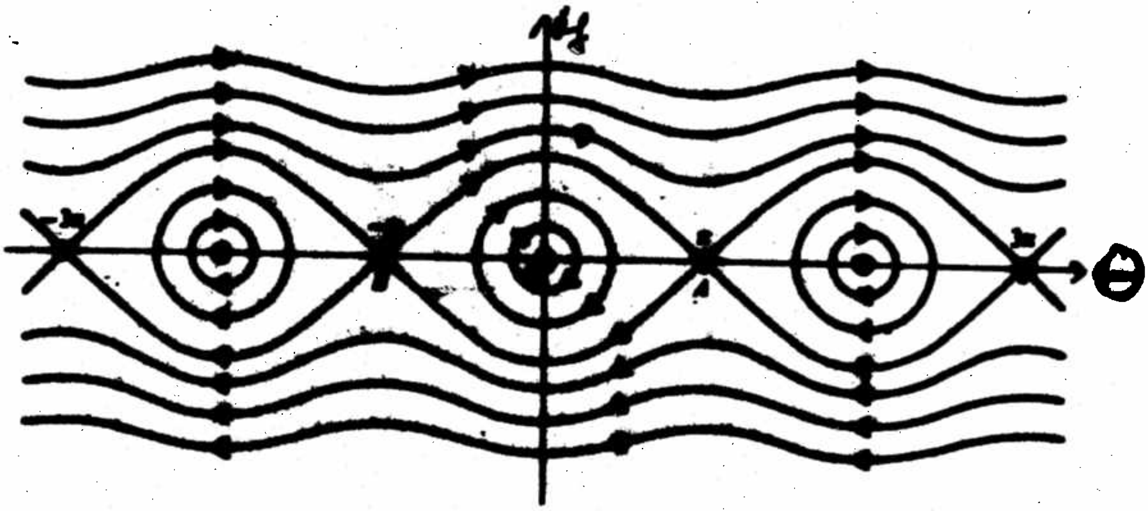
Example 5: The pendlum equation

$$ma^2 \ddot{\theta} + mga \sin\theta = 0$$

can be written as

$$\begin{cases} \dot{\theta} = y \\ \dot{y} = -\frac{g}{m} \sin\theta \end{cases}$$

Equilibrium points: $(n\pi, 0)$, $n \in \mathbb{Z}$.



[illegible] Phase diagram for the simple pendulum

e.g. $(\theta, y) = (0, 0)$ is a stable equilibrium point

$(\theta, y) = (\pi, 0)$ is an unstable equilibrium point

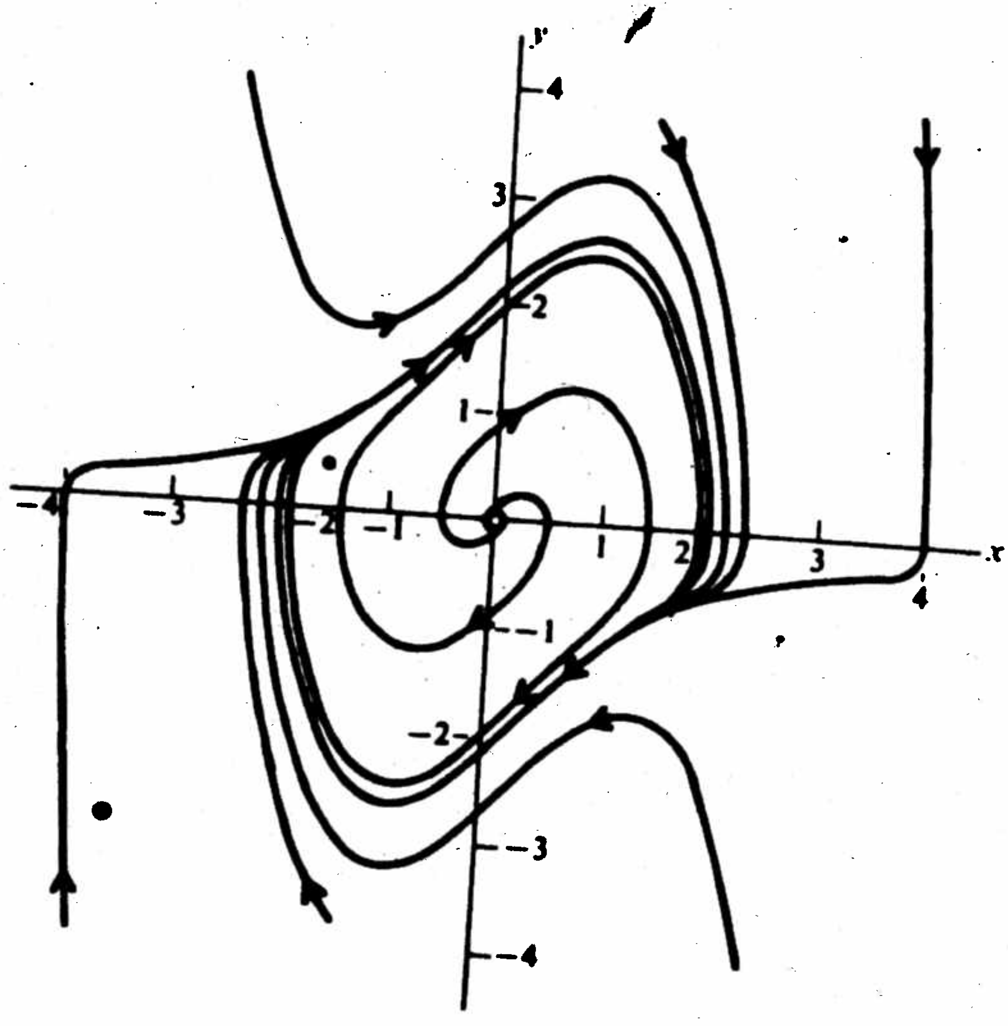
Example 6: Van der Pol's equation

$$\ddot{x} + (x^2-1)\dot{x} + x = 0$$

can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = (1-x^2)y - x \end{cases}$$

Here we have an equilibrium point $(0,0)$ and an illustration of the phase portrait indicates that we also have a limit cycle.



[redacted] Limit cycle for van der Pol's equation $\ddot{x}+(x^2-1)\dot{x}+x=0$

This limit circle is stable; All nereby solutions drift toward the limit cycle..

Example 7: A predator-prey problem (Volterra's model)

In a lake there are two species of fish: A, which lives on plants of which there is a plentiful supply and B (the predator) which subsists by eating A (the prey):

Volterra's model:

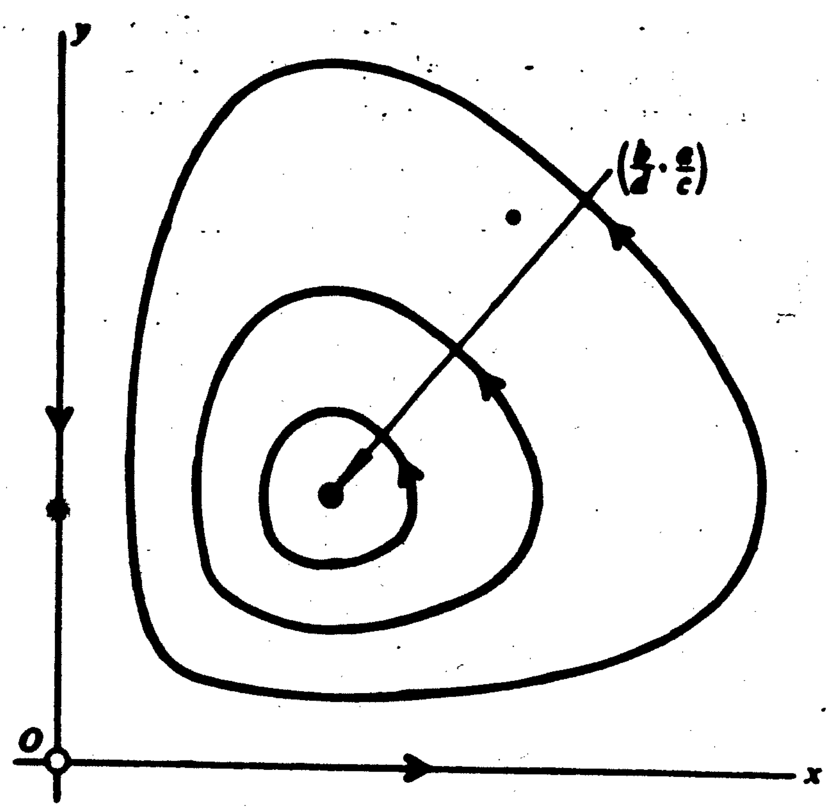
$$\begin{cases} \dot{x} = ax - cxy, & (1) \\ \dot{y} = -by + dxy, & (2) \end{cases}$$

where $x = x(t)$ is the population of A and $y = y(t)$ is the population of B at time t.

* (1) is a balance equation for the amount of prey

$$dx = ax\,dt - cxy\,dt$$

* $ax\,dt$ is the population increase in time dt due to births and "natural" deaths
* $-cxy\,dt$ is the population decrease in time dt depending on the fact that B eats A.
* In the same way we find that (2) can be interpreted as a balance equation for the amount of predator.
* Obviously the only equilibrium points are $(0,0)$ and $\left(\frac{b}{d}, \frac{a}{c}\right)$.



[illegible] Typical phase diagram for the predator–prey problem

Interprete the phase diagram!

This model is useful to explain many well-known biological phenomena when we have some "competition" between different kinds of animals e.g. between foxes and rabbits.

7 Classification of critical points.

An equilibrium point of (*) is said to be isolated if there is a neighborhood of the critical point that contains no other critical points. There are four types of isolated critical points that occur frequently namely center, node, saddle and spiral

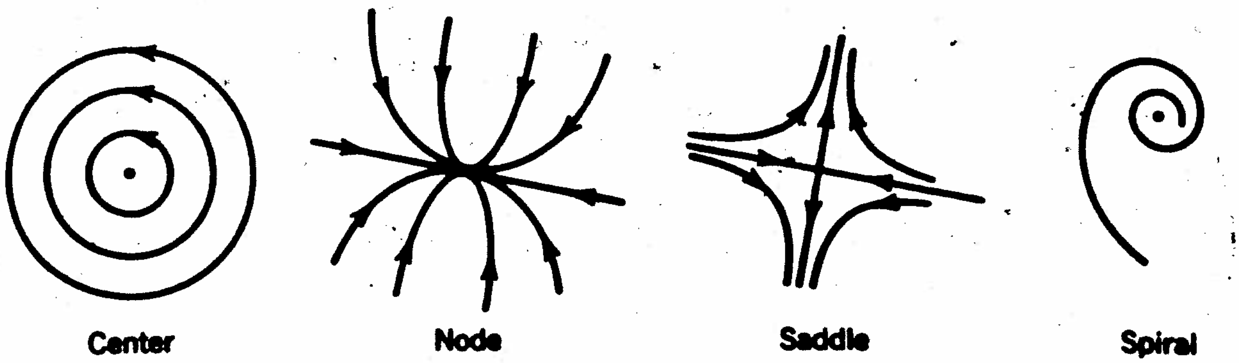


Figure 6.30. Generic critical points.

In some nonlinear systems we can have even mixed situations with " higher order critical points" se e.g. the following example:

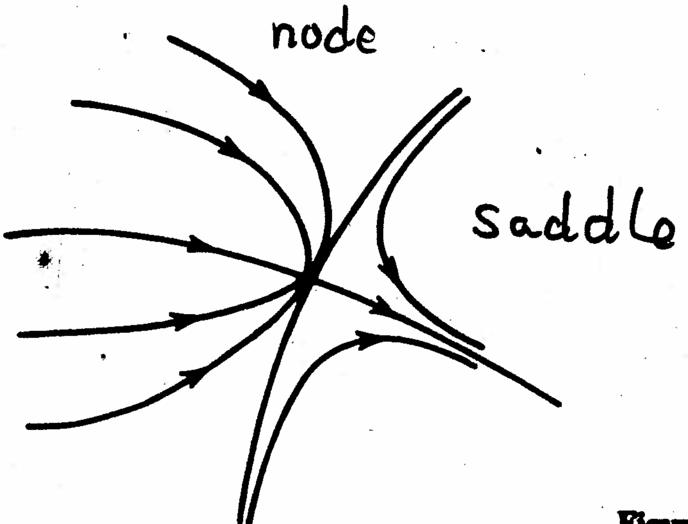


Figure 6.31. Higher order critical point.

An equilibrium point can be stable, asymptotically

8. The general solution of a linear system

$$(*) \quad \begin{cases} \dot{x} = ax + by, \\ \dot{y} = cx + dy. \end{cases}$$

$$\left(\dot{\underline{x}} = A\underline{x} \;;\; \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix},\; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

There are solutions of the form

$$x(t) = s e^{\lambda t}, \quad y(t) = r e^{\lambda t}.$$

We get

$$\begin{cases} s\lambda e^{\lambda t} = sa e^{\lambda t} + rb e^{\lambda t}, \\ r\lambda e^{\lambda t} = sc e^{\lambda t} + rd e^{\lambda t}, \end{cases}$$

so that

$$\begin{cases} (a-\lambda) s + b r = 0, \\ c s + (d-\lambda) r = 0. \end{cases}$$

Non-trivial solutions exists exactly if

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

i.e. if

$$\boxed{\lambda^2 - (a+d)\lambda + (ad-bc) = 0.}$$

[illegible] equation.

Assume that TCE has solutions λ_1, λ_2, $\lambda_1 \neq \lambda_2$ (eigenvalues) with the corresponding solutions $\begin{pmatrix} s_1 \\ r_1 \end{pmatrix}$ and $\begin{pmatrix} s_2 \\ r_2 \end{pmatrix}$, respectively (eigenvectors)

The general solution of (*) is

$$\underline{x} = C_1 \begin{pmatrix} s_1 \\ r_1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} s_2 \\ r_2 \end{pmatrix} e^{\lambda_2 t}$$

i.e.

$$\begin{cases} x(t) = C_1 s_1 e^{\lambda_1 t} + C_2 s_2 e^{\lambda_2 t} \\ y(t) = C_1 r_1 e^{\lambda_1 t} + C_2 r_2 e^{\lambda_2 t} \end{cases}$$

Example 8: Assume that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in (*) has Eigenvalues with Eigenvectors:

2	$u \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
-1	$u \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Then (*) has the general solution

$$\overline{x}(t) = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-1 \cdot t}$$

i.e.

$$\begin{cases} x(t) = 2C_1 e^{2t} - 2C_2 e^{-t}, \\ y(t) = 3C_1 e^{2t} + C_2 e^{-t} . \end{cases}$$

Example 9: Solve

$$(*) \quad \begin{cases} \dot{x} = x + y \\ \dot{y} = -5x - 3y \end{cases}$$

Sol. We consider the matrix $\begin{pmatrix} 1-\lambda & 1 \\ -5 & -3-\lambda \end{pmatrix}$ with the corresponding characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 \\ -5 & -3-\lambda \end{vmatrix} = 0$$

i.e.

$$\lambda^2 + 2\lambda + 2 = 0.$$

Eigenvalues	Corresponding eigenvector
$\lambda_1 = -1+i$	$S_1 \begin{pmatrix} 1 \\ -2+i \end{pmatrix}$
$\lambda_2 = -1-i$	$S_2 \begin{pmatrix} 1 \\ -2-i \end{pmatrix}$

Thus $(*)$ has the general solution

$$\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ -2+i \end{pmatrix} e^{(-1+i)t} + C_2 \begin{pmatrix} 1 \\ -2-i \end{pmatrix} e^{(-1-i)t}$$

i.e.

$$(1) \quad \begin{cases} x(t) = C_1 e^{(-1+i)t} + C_2 e^{(-1-i)t} \\ y(t) = C_1(-2+i) e^{(-1+i)t} + C_2(-2-i) e^{(-1-i)t} \end{cases}$$

(1) gives all <u>complex</u> solutions. The real solutions are

$$\begin{cases} x(t) = e^{-t}(d_1 \cos t + d_2 \sin t) \\ y(t) = -e^{-t}\{(2d_1 - d_2)\cos t + (d_1 + 2d_2)\sin t\} \end{cases}$$

$$x(t) = c_1 e^{-t} \cdot e^{it} + c_2 e^{-t} e^{-it} =$$

$$e^{-t}\left(c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t)\right) =$$

$$= e^{-t}\left(\underbrace{(c_1+c_2)}_{= d_1} \cos t + \underbrace{i(c_1 - c_2)}_{d_2} \sin t\right)$$

$$y(t) = c_1(-2+i) e^{-t} e^{it} + c_2(-2-i) e^{-t} e^{-it} =$$

$$e^{-t}\left(c_1(-2+i)(\cos t + i\sin t) + c_2(-2-i)(\cos t - i\sin t)\right)$$

$$= e^{-t}\left(-2\underbrace{(c_1+c_2)}_{d_1} + \underbrace{i(c_1-c_2)}_{d_2}\right)\cos t + \underbrace{\left(-2\underbrace{i(c_1-c_2)}_{d_2} + (-1)\overbrace{(c_1+c_2)}^{d_1}\right)\sin t}$$

$\therefore$

$$\begin{cases} x(t) = e^{-t}(d_1 \cos t + d_2 \sin t) \\ y(t) = e^{-t}((-2d_1 + d_2)\cos t + (-2d_2 - d_1)\sin t) \end{cases}$$

Real constants d_1 and d_2 ?

$$c_1 = a + ib$$

$$c_2 = c + id$$

$$d_1 = c_1 + c_2 = a + c + i(b+d)$$

$$d_2 = i(c_1 - c_2) = (-b+d) + i(a-c)$$

Conclusion: The constants d_1 and d_2 are real iff $a = c$ and $b = -d$ i.e. if $c_1 = \overline{c_2}$.

9. Classification of the equilibrium point (0,0) of the system

$$(*)\begin{cases}\dot{x} = ax + by\\ \dot{y} = cx + dy\end{cases} \Leftrightarrow \dot{\underline{x}} = A\underline{x}\ ,\ \underline{x} = \begin{pmatrix}x\\y\end{pmatrix}$$

1° $0 < \lambda_1 < \lambda_2$. Let $\begin{pmatrix}s_1\\r_1\end{pmatrix}$ and $\begin{pmatrix}s_2\\r_2\end{pmatrix}$ be eigenvectors corresponding to λ_1 and λ_2, respectively. Then the general solution of (*) is given by

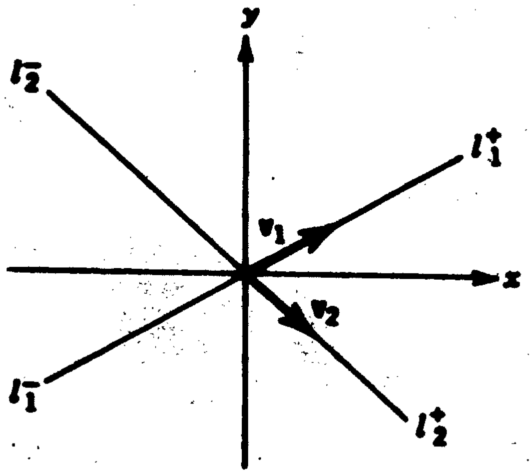
$$(**)\qquad \underline{x}(t) = c_1\underbrace{\begin{pmatrix}s_1\\r_1\end{pmatrix}}_{v_1}e^{\lambda_1 t} + c_2\underbrace{\begin{pmatrix}s_2\\r_2\end{pmatrix}}_{v_2}e^{\lambda_2 t}.$$

$c_2=0,\ c_1>0$: $\underline{x}(t)$ is a path on l_1^+. →

$c_2=0,\ c_1<0$: $\underline{x}(t)$ is a path on l_1^-. →

$c_1=0,\ c_2>0$: $\underline{x}(t)$ is a path on l_2^+. →

$c_1=0,\ c_2<0$: $\underline{x}(t)$ is a path on l_2^-. →



$v_1 = \begin{pmatrix}s_1\\r_1\end{pmatrix}$

$v_2 = \begin{pmatrix}s_2\\r_2\end{pmatrix}$

$c_1 \neq 0, c_2 \neq 0$

"$t \to -\infty$" Then $e^{\lambda_1 t} >> e^{\lambda_2 t}$ and, thus,

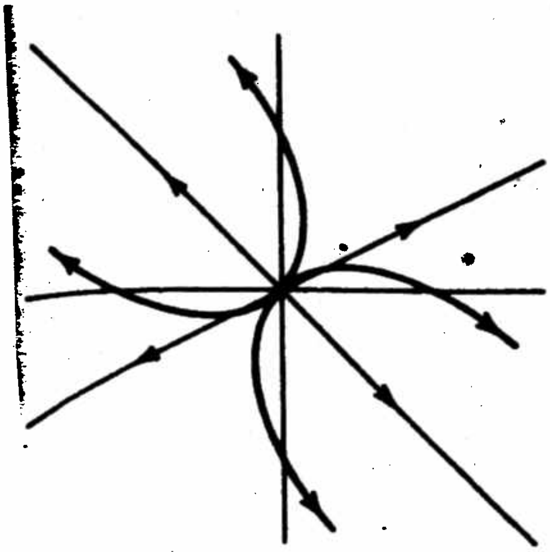
$\underline{x} \approx c_1\begin{pmatrix}s_1\\r_1\end{pmatrix}e^{\lambda_1 t}$ for large negative t

In particular $\underline{x} \to \begin{pmatrix}0\\0\end{pmatrix}$ as $t \to -\infty$

"$t \to +\infty$" Then

$$\underline{x} \approx c_2 \begin{pmatrix} s_2 \\ r_2 \end{pmatrix} e^{\lambda_2 t} \quad \text{for large positive } t$$

Thus $\underline{x}$ tends to infinity with a slope asymptotic to $\begin{pmatrix} s_2 \\ r_2 \end{pmatrix}$ as $t \to \infty$.



[redacted] Unstable node.

Thus $(0,0)$ is an <u>unstable node</u>. The lines defined by the eigenvectors $\begin{pmatrix} s_1 \\ r_1 \end{pmatrix}$ and $\begin{pmatrix} s_2 \\ r_2 \end{pmatrix}$ are called <u>separatrices</u>.

2° <u>$\lambda_2 < \lambda_1 < 0$</u>. In the same way as above we find that $(0,0)$ is <u>a stable node</u>. The phase portrait is the same but with reversed arrows.

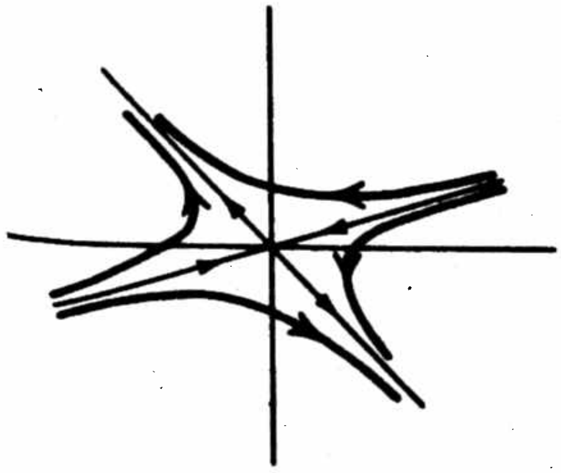
3° <u>$\lambda_1 < 0 < \lambda_2$</u>.

$c_2 = 0,\ c_1 > 0$: $\underline{x}(t)$ is a path on l_1^+ $\leftarrow$

$c_2 = 0,\ c_1 < 0$: $\underline{x}(t)$ is a path on l_1^- $\leftarrow$

$c_1 = 0,\ c_2 > 0$: $\underline{x}(t)$ is a path on l_2^+ $\rightarrow$

$c_1 = 0,\ c_2 < 0$: $\underline{x}(t)$ is a path on l_2^- $\rightarrow$



$c_1 > 0$
$c_2 > 0$

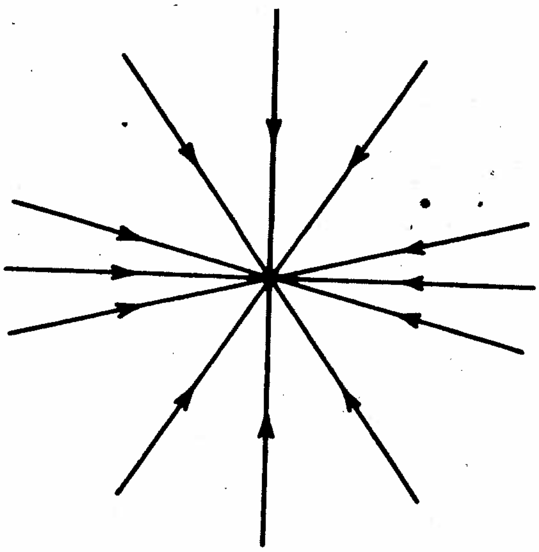
[redacted] Saddle point.

4° $\underline{\lambda_1 = \lambda_2 = \lambda < 0}$

(a) There exists two linearly independent eigenvectors $\binom{s_1}{r_1}$ and $\binom{s_2}{r_2}$. Then the solution of (*) is

$$\underline{x}(t) = c_1 \binom{s_1}{r_1} e^{\lambda t} + c_2 \binom{s_2}{r_2} e^{\lambda t} = \binom{a_1}{a_2} e^{\lambda t},$$

where a_1 and a_2 are arbitrary!

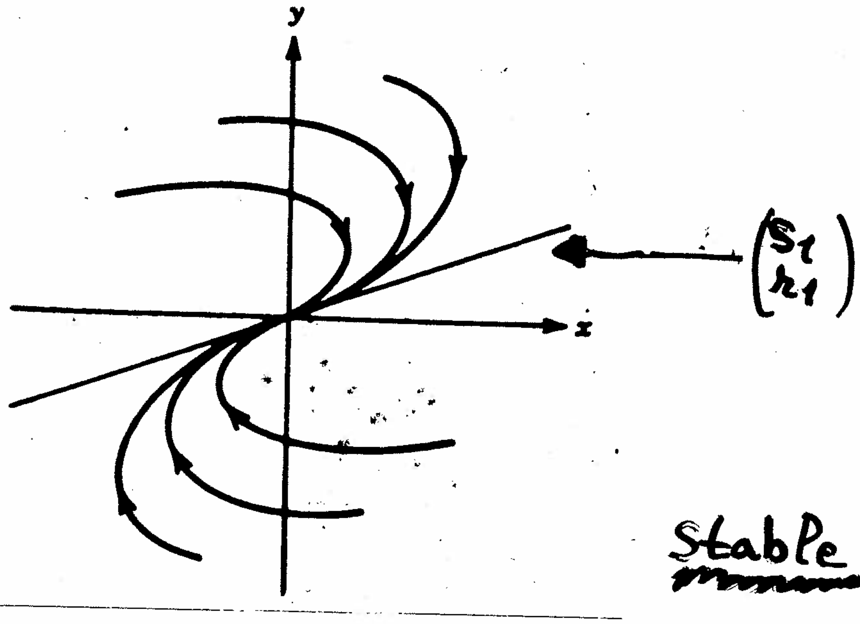


[redacted] Stable node.

(b) There exists only one eigenvector $\binom{s_1}{r_1}$. Then the solution of (*) is of the form

$$\underline{x}(t) = \left(c_1 \binom{s_1}{r_1} + c_2 \left(\binom{s_2}{r_2} + t \binom{s_1}{r_1} \right) \right) e^{\lambda t}$$

For large t we have $\underline{x}(t) \approx c_2 t \binom{s_1}{r_1} e^{\lambda t}$.



Stable node

5° $\lambda_1 = \lambda_2 = \lambda > 0$

This case is exactly the same as case 4° with the directions reversed; $(0,0)$ is an unstable node

6° $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

By using the same arguments as in our Example 9 we find that the real solutions of $(*)$ is

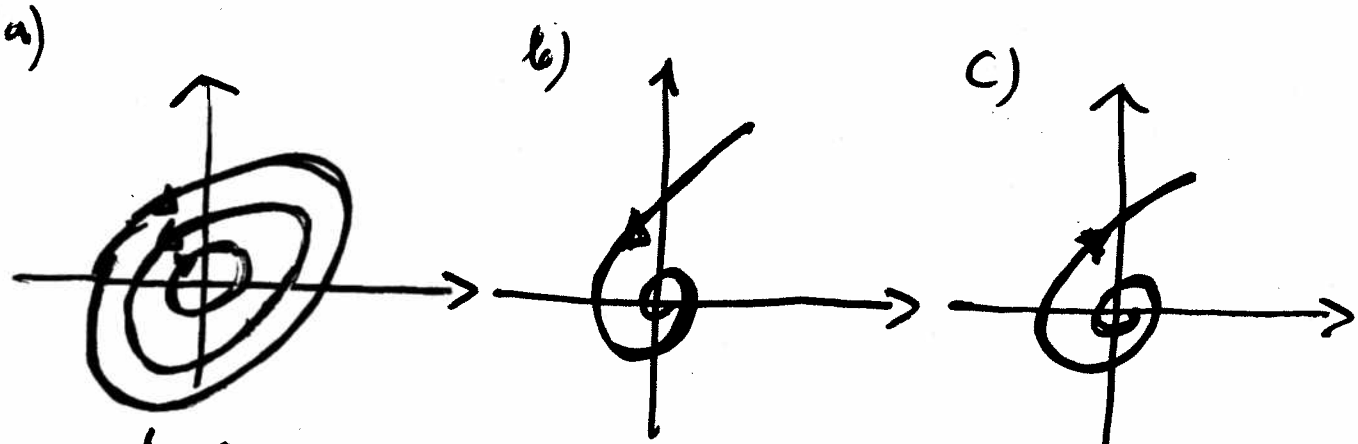
$$\underline{X}(t) = c_1 e^{\alpha t}\left(\begin{pmatrix} s_1 \\ r_1 \end{pmatrix}\cos\beta t - \begin{pmatrix} s_2 \\ r_2 \end{pmatrix}\sin\beta t\right) +$$

$$+ c_2 e^{\alpha t}\left(\begin{pmatrix} s_1 \\ r_1 \end{pmatrix}\sin\beta t + \begin{pmatrix} s_2 \\ r_2 \end{pmatrix}\cos\beta t\right)$$

$\alpha = 0$ Then $x(t)$ and $y(t)$ are periodic with period $2\pi/\beta$. There fore $(0,0)$ is a center.

) $\alpha < 0$ Then the amplitude of $\underline{X}$ decreases and we have stable spiralas.

) $\alpha > 0$ Then the amplitude of $\underline{X}$ increases



$\alpha = 0$ center | $\alpha < 0$ stable spiral | $\alpha > 0$ unstable spiral

Example 10: Consider

$$\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 2x - 2y \end{cases} \Leftrightarrow \underline{\dot{x}} = A\underline{x}$$

The coefficient matrix $A \doteq \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has the characteristic equation

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

and, thus, the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$

This is Case 3° and we conclude that $(0,0)$ is an unstable saddle point.

The eigenvectors are found by solving the linear system

$$\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}\begin{pmatrix} s \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = \lambda_1 = -1}$ implies that

$$\begin{cases} 4s - 2r = 0 \\ 2s - r = 0 \end{cases}$$

$\therefore$ A corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = \lambda_2 = 2$ implies that

$$\begin{cases} s - 2r = 0 \\ 2s - 4r = 0 \end{cases}$$

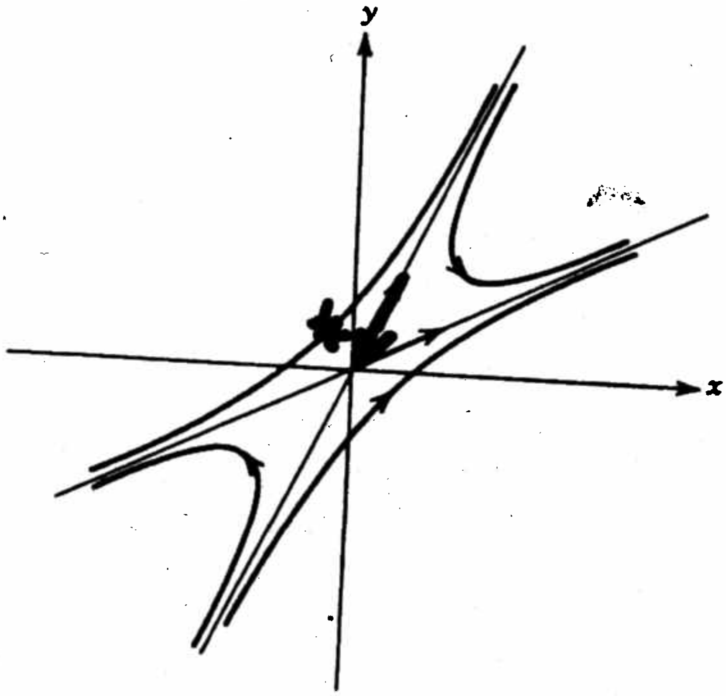
$\therefore$ A corresponding eigenvector is $\binom{2}{1}$

The general solution of the system is

$$\underline{x} = c_1 \binom{1}{2} e^{-t} + c_2 \binom{2}{1} e^{2t} \quad \text{i.e.}$$

$$\begin{cases} x(t) = c_1 e^{-t} + 2c_2 e^{2t}, \\ y(t) = 2c_1 e^{-t} + c_2 e^{2t}. \end{cases}$$

The eigenvectors defines the directions of the separatrices.



Example 11: Consider

$$(*) \begin{cases} \dot{x} = 3x - 2y - 4 \\ \dot{y} = 2x - 2y - 2 \end{cases}$$

Here the critical point is $(2,1)$. Moreover $(*)$ can be written as

$$\begin{cases} \dot{x}_1 = 3x_1 - 2y_1, \\ \dot{y}_1 = 2x_1 - 2y_1, \end{cases}$$

where $x_1 = x-2$ and $y_1 = y-1$. By using the result from Example 10 we find that $(2,1)$ is an unstable saddle, that the solution is

$$\begin{cases} x(t) = 2 + c_1 e^{-t} + 2c_2 e^{2t}, \\ y(t) = 1 + 2c_1 e^{-t} + c_2 e^{2t}, \end{cases}$$

and that phase portrait is that in Example 10 moved 2 units in the x-direction and 1 unit in the y-direction. ■

In a similar way as in Example 11 we can instead of $(*)$ study the more general

$$(**) \quad \begin{cases} \dot{x} = a(x-\alpha_0) + b(y-\beta_0), \\ \dot{y} = c(x-\alpha_0) + d(y-\beta_0). \end{cases}$$

The equilibrium point is here equal to (α_0, β_0). By making the transformations $x_1 = x-\alpha_0$ and $y_1 = y-\alpha_1$ we can carry over $(**)$ to the situation studied in $(*)$ (with the equilibrium point $(0,0)$).

[redacted]:

Problems - Lecture 9

1. Sketch the phase portraits for the following systems:

(a) $\begin{cases} \dot{x} = x - 3y \\ \dot{y} = -3x + y. \end{cases}$

(b) $\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y. \end{cases}$

2.* (a) Solve the follow dynamical system:

(*) $\begin{cases} \dot{x} = x + y \\ \dot{y} = 4x - 2y. \end{cases}$

(b) Sketch the phase portrait of (*).

3.* The equation for a damped harmonic oscillator is

$$m\ddot{x} + a\dot{x} + kx = 0, \quad m, a, k > 0.$$

Write the equation as a dynamical system by introducing $y = \dot{x}$ and show that $(0,0)$ is a critical point. Describe the nature and stability of the critical point in the following cases:

(a) $a = 0$ (b) $a^2 - 4km = 0$

(c) $a^2 - 4km < 0$ (d) $a^2 - 4km > 0$.

4. Find the values of μ where the solutio[n] to the following system bifurcate and examine the stability of the origin in each case:

(a) $\begin{cases} \dot{x} = x + \mu y \\ \dot{y} = \mu x + y . \end{cases}$ (b) $\begin{cases} \dot{x} = y \\ \dot{y} = -2x + \mu y . \end{cases}$

(c) $\begin{cases} \dot{x} = x + y \\ \dot{y} = \mu x + y . \end{cases}$ (d) $\begin{cases} \dot{x} = \mu y + xy \\ \dot{y} = -\mu x + \mu y + x^2 + 2xy . \end{cases}$

5.* (a) Describe the Verhult's population model. In particular, describe how this model can be used to illustrate the concepts attractor and chaos. What is FeigenBaum's constant?

(b) Describe how you can illustrate Julia sets.
What is the MandelBrot set?

6. Determine whether the following system[s] admit periodic solutions:

(a) $\begin{cases} \dot{x} = y \\ \dot{y} = (x^2 + 1)y - x^5 . \end{cases}$ (b) $\begin{cases} \dot{x} = y \\ \dot{y} = y^2 + x^2 + 1 . \end{cases}$

(c) $\begin{cases} \dot{x} = y \\ \dot{y} = 3x^2 - y - y^5 . \end{cases}$