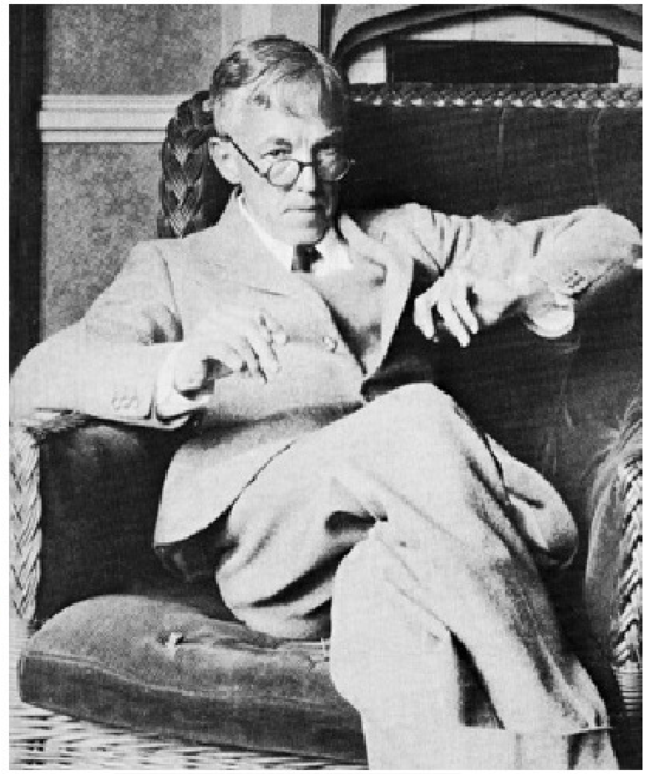
My life with Hardy and his inequalities

Lars-Erik Persson
From1944 -

by
Lars-Erik Persson

UiT, The Arctic University of Norway and
Uppsala University
Sweden



Godfrey Harold Hardy
1877-1947

IWOTA 2019, Lisbon, Portugal

[1] A.Kufner, L.E. Persson and N. Samko, Weighted Inequalities of Hardy type, World Scientific, Second edition, New Jersey-London-etc., 2017

[2] L.E. Persson, Lecture Notes, Collège de France, Pierre-Louis Lions' Seminar, November 2015.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1 The prehistory of the Hardy inequality

We consider the following statements of the Hardy inequality: the discrete inequality asserts that if $\{a_n\}_1^\infty$ is a sequence of non-negative real numbers then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p, \quad p>1, \tag{1.1}$$

The continuous inequality informs us that if f is a non-negative p-integrable function on $(0,\infty)$, then f is integrable over the interval $(0,x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1. \tag{1.2}$$

The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history or, as we call it, prehistory. Contributions of mathematicians other than G.H.Hardy, such as E.Landau, G.Pòlya, E.Schur and M.Riesz, are important here.

This prehistory was described in detail in:

$[*]$ **A.Kufner, L.Maligranda and L.E.Persson. The prehistory of the Hardy inequality, Amer. Math. Monthly, 113(8):715–732, 2006**

In particular, the following is clear:

(a) Inequalities (1.1) **and** (1.2) **are the standard forms of the Hardy inequalities that can be found in many text books on Analysis and were highlighted first in the famous book *Inequalities* by Hardy, Littlewood and Pólya.**

(b) By restricting (1.2) **to the class of step functions one proves easily that** (1.1) **implies** (1.2)**.**

(c) The constant $(p/(p-1))^p$ **in both** (1.1) **and** (1.2) **is *sharp*: it cannot be replaced with a smaller number such that** (1.1) **and** (1.2) **remain true for all relevant sequences and functions, respectively.**

(d) The main motivation for Hardy to begin this dramatic history in 1915 was to find a simpler proof of the Hilbert inequality from 1906:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi \left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2} \tag{1.3}$$

(In Hilbert's version of (1.3) **the constant** 2π **appears instead of the sharp one** π**.) We remark that nowadays the following more general form of** (1.3) **is also sometimes referred in the literature as Hilbert's inequality**

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.4}$$

where $p > 1$ **and** $p' = p/(p-1)$. **However, Hilbert was not even close to consider this case (the** l_p**-spaces appeared only in 1910).**

(e) The first weighted version of (1.2) **was proved by Hardy himself in 1928:**

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^{\infty} f^p(x)x^a dx, \tag{1.5}$$

where f **is a measurable and non-negative function on** $(0,\infty)$ **whenever** $a < p-1, p > 1$**.**

1.1 A new look on the inequalities (1.1) and (1.5)

Observation 1.1. *We note that for $p > 1$*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$
$$\Leftrightarrow$$
$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1\cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{1.6}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$.

This means that Hardy's inequality (1.2) is equivalent to (1.6) for $p > 1$ and, thus, that Hardy's inequality can be proved in the following simple way (see form (1.6)): By Jensen's inequality and Fubini's theorem we have that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x} =$$
$$\int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

By instead making the substitution $f(t) = g(t^{\frac{p-1-a}{p}})t^{-\frac{1+a}{p}}$ in (1.5) we see that also this inequality is equivalent to (1.6). These facts imply especially the following:

(a) Hardy's inequalities (1.1) **and** (1.5) **hold also for** $p < 0$ **(because the function** $\varphi(u) = u^p$ **is convex also for** $p < 0$**) and hold in the reverse direction for** $0 < p < 1$ **(with sharp constants** $\left(\frac{p}{1-p}\right)^p$ **and** $\left(\frac{p}{a+1-p}\right)^p$, $a > p - 1$, **respectively).**

(b) The inequalities (1.1) **and** (1.5) **are equivalent.**

(c) The inequality (1.6) **holds also for** $p = 1$ **which gives us a possibility to interpolate and get more information about the mapping properties of the Hardy operator. In particular, we can use interpolation theory to see that in fact the Hardy operator** H **maps each interpolation space** B **between** $L_1\left((0,\infty), \frac{dx}{x}\right)$ **and** $L_\infty\left((0,\infty), \frac{dx}{x}\right)$ **into** B, **i.e. that** $\|Hf\|_B \leq C\|f\|_B$.

2 On the further development of Hardy type inequalities

Some parts of this development are described in the books:

[A] A.Kufner, L.E. Persson and N. Samko, Weighted Inequalities of Hardy type, World Scientific, Second edition, New Jersey-London-etc., 2017.

[B] A.Kufner, L.Maligranda and L.E.Persson, The Hardy Inequality. About its History and Some Related Results, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[C] V.Kokilashvili, A.Meskhi and L.E.Persson, Weighted Norm Inequalities for Integral Transforms with Product Weights, Nova Scientific Publishers, Inc., New York, 2010.

One important early question was the following:
For which weights u and v does it hold that

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p},$$

$0 < b \leq \infty$, **for some finite constant** C **?**

During the last 80 years it has been a lot of activities to answer this and more general questions concerning Hardy type inequalities and a lot of interesting results have been developed.

Just as one example we mention the following well known result:

Theorem 2.1. ***Let*** $1 < p \leq q < \infty$ ***and*** u ***and*** v ***be weight functions on*** $\mathbb{R}_+$***. Then each of the following conditions are necessary and sufficient for the inequality***

$$\left(\int_0^b \left(\int_0^x f(t)\, dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_0^b f^p(x) v(x)\, dx \right)^{\frac{1}{p}} \tag{2.1}$$

to hold for all positive and measurable functions on $\mathbb{R}_+$***:***
a) the Muckenhoupt-Bradley-type condition,

$$A_{MB} := \sup_{x>0} \left(\int_x^b u(t)\, dt \right)^{\frac{1}{q}} \left(\int_0^x v(t)^{1-p'}\, dt \right)^{\frac{1}{p'}} < \infty, \tag{2.2}$$

with the estimation $C \in [A_{MB}, \lambda A_{MB}]$ ***for the best constant*** C ***in*** (2.1)***, where***

$$\lambda = min(p^{1/q}(p')^{1/p'}, q^{1/q}(q')^{1/p'}).$$

b) The condition

$$A_{PS} := \sup_{x>0} V(x)^{-\frac{1}{p}} \left(\int_0^x u(t) V(t)^q\, dt \right)^{\frac{1}{q}} < \infty, \quad V(x) := \int_0^x v(t)^{1-p'}\, dt, \tag{2.3}$$

with $C \in [A_{PS}, p' A_{PS}]$ ***for the best constant in*** (2.1)***.***

Remark 2.2. **The dramatic history until** (2.2) **was derived can be found in the book [B]. A simple proof of the characterization** (2.2) **was given by B.Muckenhoupt in 1972 for** $p = q$ **and by J.S.Bradley in 1978 for** $p \leq q$. **In 2002 L.E. Persson and V.D. Stepanov presented an elementary proof of the alternative condition** (2.3)**.**

Remark 2.3. **It has recently been discovered that also these two conditions to characterize** (2.1) **are not unique and can even be replaced by infinite many equivalent conditions, in fact even by scales of conditions. In Section 3.5 of this lecture also this result will be presented.**

3 Examples of complementary and newer results

3.1 Further consequences of the new look presented in Section 1.1

For the finite interval case we need the following extention of our basic observation in Section 1.1.

Lemma 3.1. ***Let*** g ***be a non-negative and measurable function on*** $(0,\ell), 0<\ell\leq\infty$.

a) ***If*** $p<0$ ***or*** $p\geq 1$, ***then***

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1\cdot\int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{3.1}$$

(In the case $p<0$ ***we assume that*** $g(x)>0, 0<x\leq\ell$***).***

b) ***If*** $0<p\leq 1$, ***then*** (3.1) ***holds in the reversed direction.***

c) The constant $C=1$ *is sharp in both a) and b).*

By using this Lemma and straightforward calculations the following statement can be proved, see

$[*]$ **L.E.Persson and N. Samko, What should have happened if Hardy discovered this?, J. Inequal. Appl. Springer Open 2012, 2012:29.**

Theorem 3.2. ***Let*** $0 < \ell \leq \infty$, ***let*** $p \in \mathbb{R}_+ \setminus \{0\}$ ***and let*** f ***be a non-negative function. Then a) the inequality***

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq$$

$$\left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a \left[1 - \left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right] dx \tag{3.2}$$

holds for all measurable functions f, ***each*** $\ell, 0 < \ell \leq \infty$ ***and all*** a ***in the following cases:***

$$(a_1) \qquad p \geq 1, a < p-1,$$

$$(a_2) \qquad p < 0, a > p-1.$$

b) For the case $0 < p < 1, a < p-1$, ***inequality*** (3.2) ***holds in the reversed direction under the conditions considered in a).***

c) The inequality

$$\int_\ell^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{a_0} dx \leq$$

$$\left(\frac{p}{a_0+1-p}\right)^p \int_\ell^\infty f^p(x)x^{a_0} \left[1 - \left(\frac{\ell}{x}\right)^{\frac{a_0+1-p}{p}}\right] dx \tag{3.3}$$

holds for all measurable functions f, ***each*** $\ell, 0 \leq \ell < \infty$ ***and all*** a ***in the following cases:***

$$(c_1) \qquad p \geq 1, a_0 > p-1,$$

$$(c_2) \qquad p < 0, a_0 < p-1.$$

d) For the case $0 < p \leq 1$, ***inequality*** (3.3) ***holds in the reversed direction***

under the conditions considered in c).

e) All inequalities above are sharp.

f) Let $p \geq 1$ ***or*** $p < 0$. ***Then, the statements in a) and c) are equivalent for all permitted*** a ***and*** a_0 ***because they are in all cases equivalent to*** (3.1) ***via substitutions.***

g) Let $0 < p < 1$. ***Then, the statements in b) and d) are equivalent for all permitted*** a ***and*** a_0.

3.2 A further development of Bennett's inequalities with two sharp constants

There exists very few Hardy type inequalities with sharp constant in the limit case (formally corresponding to when $a = p-1$ in (3.2)) and when the interval $(0,\infty)$ is replaced by a finite interval $(0,\ell)$, $\ell < \infty$. We continue by giving two such examples (Bennett's inequalities from 1973), which have direct applications e.g. to Interpolation Theory.

Proposition A: Let $\alpha > 0, 1 \leq p \leq \infty$ and f be a non-negative and measurable function on $[0,1]$. Then

$$\left(\int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} \leq$$

$$\alpha^{-1}\left(\int_0^1 x^p[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p}, \tag{3.4}$$

and

$$\left(\int_0^1 [\log(e/x)]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} \leq$$

$$\alpha^{-1}\left(\int_0^1 x^p[\log(e/x)]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p} \tag{3.5}$$

with the usual modification if $p = \infty$.

The next refinements of the inequalities (3.4) and (3.5) in Proposition A was proved in 2014 in

[$*$] S. Barza, L.E. Persson and N. Samko, Some new limit Hardy-type inequalities via convexity, J.Inequal.Appl. 2014, 2014:6.

Theorem 3.3. ***Let*** $\alpha, p > 0$ ***and*** f ***be a non-negative and measurable function on*** $[0,1]$.
(a) ***If*** $p > 1$, ***then***

$$\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p +$$
$$\alpha^p \int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \leq$$
$$\leq \int_0^1 x^p [\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \tag{3.6}$$

and

$$\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p +$$
$$\alpha^p \int_0^1 [\log(e/x)]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p \frac{dx}{x} \leq$$
$$\leq \int_0^1 x^p [\log(e/x)]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{3.7}$$

Both constants α^{p-1} ***and*** α^p ***in*** (3.6) ***and*** (3.7) ***are sharp. Equality is never attained unless*** f ***is identically zero.***
(b) ***If*** $0 < p < 1$, ***then both*** (3.6) ***and*** (3.7) ***hold in the reverse direction and the constants in both inequalities are sharp. Equality is never attained unless*** f ***is identically zero.***
(c) ***If*** $p = 1$ ***we have equality in*** (3.6) ***and*** (3.7) ***for any measurable function*** f ***and any*** $\alpha > 0$.

3.3 The sharp constant for the power weighted case when $1 < p < q$

By applying the general results (see Theorem 2.1 and the corresponding dual result) for the power weighted case we get the following:

Example 3.4. **The inequality**

$$\left(\int_0^\infty \left(\int_0^x f(t)dt \right)^q x^\alpha dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(x) x^\beta dx \right)^{\frac{1}{p}}$$

holds for $1 < p \leq q < \infty$ **, if and only if**

$$\beta < p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1.$$

Example 3.5. **The inequality**

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt \right)^q x^\alpha dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(x) x^\beta dx \right)^{\frac{1}{p}}$$

holds for $1 < p \leq q < \infty$ **, if and only if**

$$\beta > p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1.$$

For the next result we need the following Lemma.

Lemma 3.6. ***Let*** $1 < p < q < \infty$. ***The following statements (a) and (b) hold and are equivalent:***
(a) The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{1/q} \leq C \left(\int_0^x f^p(x)x^\beta dx\right)^{1/p} \tag{3.8}$$

holds for all measurable functions $f(t)$ ***on*** $(0,\infty)$ ***if and only if***

$$\beta < p-1 \text{ and } \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1. \tag{3.9}$$

(b) The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^{\alpha_0} dx\right)^{1/q} \leq C \left(\int_0^\infty f^p(x)x^{\beta_0} dx\right)^{1/p} \tag{3.10}$$

holds for all measurable functions $f(t)$ ***on*** $(0,\infty)$ ***if and only if***

$$\beta_0 > p-1, \frac{\alpha_0+1}{q} = \frac{\beta_0+1}{p} - 1. \tag{3.11}$$

Moreover, it yields that
(c) the formal relation between the parameters β ***and*** β_0 ***is*** $\beta_0 = -\beta - 2 + 2p$ ***and in this case the best constants*** C ***in*** (3.8) ***and*** (3.10) ***are the same.***

The next result was recently proved in 2015 by L.E.Persson and S.Samko, see [∗]. Indeed, this result gave a final answer for an old open question, where G.A.Bliss in 1930 found the best constant for the case $\beta = 0$ **in** (3.8)**.**

[∗] L.E.Persson and S.Samko, A note on the best constants in some Hardy inequalities, J.Math.Inequal 9(2015), no2,437-447.

Theorem 3.7. ***Let*** $1 < p < q < \infty$ ***and the parameters*** α ***and*** β ***satisfy*** (3.9)***. Then the sharp constant in*** (3.8) ***is*** $C = C^*_{pq}$, ***where***

$$C^*_{pq} = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.12}$$

Equality in (3.8) ***occurs exactly when***

$$f(x) = \frac{cx^{-\frac{\beta}{p-1}}}{\left(dx^{\frac{p-1-\beta}{p-1}\cdot\left(\frac{q}{p}-1\right)}+1\right)^{\frac{q}{q-p}}}.$$

Moreover,

$$C^*_{pq} \to \frac{p}{p-1-\beta} \text{ as } q \to p.$$

By using this result and Lemma 3.6 we obtain the following sharp constant in (3.10)**:**

Theorem 3.8. ***The sharp constant in*** (3.10) ***with parameters satisfying*** (3.11) ***for the case*** $1 < p < q < \infty$ ***is*** $C^{\sharp}_{p,q}$, ***where*** $C^{\sharp}_{p,q}$ ***coincides with the constant*** $C^{*}_{p,q}$ ***with*** β ***replaced by*** $-\beta_0 - 2 + 2p$.
Equality in (3.10) ***occurs if and only if*** $f(x)$ ***is of the form***

$$f(x) = \frac{cx^{\beta_0/p-1}}{(dx^{(\frac{\beta_0+1-p}{p-1})(\frac{q}{p}-1)} + 1)^{\frac{q}{q-p}}} \text{ a.e..}$$

Moreover, we have the continuity between sharp constants when $q \to p$, ***i.e.***

$$C^{\sharp}_{p,q} \to \frac{p}{\beta_0 + 1 - p} \text{ as } q \to p.$$

Remark 3.9. **In the same paper also the sharp constants in the corresponding multi-dimensional Hardy type inequalities were derived.**

3.4 Concerning the kernel operator case

Here we study in particular characterizations of the following more general Hardy-type inequality

$$\|Tf\|_{q,u} \leq C\|f\|_{p,v}, \tag{3.13}$$

where u **and** v **are weight functions and**

$$Tf(x) := \int_a^x k(x,y)f(y)dx,$$

$k(x,y)$ **denote a positive kernel.**

Some facts:

(a) Without restrictions on the kernel $k(x,y)$ **the problem is open.**

(b) The solution of this problem is known for a number of special cases and parameters.

The following new result was recently proved for the general kernel operator case (in the previously mentioned review article by A.Kufner, L.E.Persson and N.Samko from 2015):

[$*$] A. Kufner, L.E.Persson and N.Samko, Hardy type inequalities with kernels: the current status and some new results, Math.Nach. 290 (2017).

Theorem 3.10. ***Let*** $1 < p \leq q < \infty,\ a < b \leq \infty$, u ***and*** v ***are weights. Let*** $k(x, y)$ ***be a non-negative kernel.***
(a) ***Then*** (3.13) ***holds if***

$$A_s := \sup_{a<y<b} \left(\int_y^b k^q(x, y)u(x)V^{(\frac{q(p-s-1)}{p})}(x)dx \right)^{1/q} V^{s/p}(y) < \infty, \tag{3.14}$$

for any $s < p - 1$.
(b) ***The condition*** (3.14) ***can not be improved in general for*** $s > 0$ ***because for product kernels it is even necessary and sufficient for*** (3.13) ***to hold.***
(c) ***For the best constant*** C ***in*** (3.13) ***we have the following estimate***

$$C \leq \inf_{s<p-1} \left(\frac{p}{p - s - 1} \right)^{1/p'} A_s.$$

Here and the sequel we use the following notations

$$U(x) := \int_x^b u(y)dy, \qquad V(x) := \int_a^x v^{1-p'}(y)dy, \tag{3.15}$$

Remark 3.11. **This result opens a possibility that the condition** (3.14) **can be a candidate to solve the open question we have pointed out in 3.4(a) above.**

Remark 3.12. **In Section 3.6 we present some multidimensional inequalities involving kernel type operators and decreasing functions (and with sharp constant in each case).**

3.5 Some new scales of conditions to characterize the modern forms of Hardy's inequality

We have recently proved that the conditions $A_{MB} < \infty$ and $A_{PS} < \infty$ in Theorem 2.1 can be replaced by infinite many equivalent conditions even by scales of conditions as presented below. We refer to a review article [$*$] by A.Kufner, L.E.Persson and N.Samko from 2013, and references therein.

[$*$] A.Kufner, L.E. Persson, and N.Samko, Some new scales of weight characterizations of Hardy-type inequalities, Operator theory, pseudo-differential equations, and mathematical physics, Oper. Theory Adv. Appl., 228:261–274, 2013.

Theorem 3.13. ***Let*** $1 < p \leq q < \infty$, $0 < s < \infty$, ***and define, for the weight functions*** u, v, ***the functions*** U ***and*** V ***by*** (3.15)***. Then*** (2.1) ***can be characterized by any of the conditions*** $A_i(s) < \infty$, ***where*** $A_i(s), i = 1, 2, 3, 4$ ***are defined by:***

$$A_1(s) := \sup_{0<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'}-s)}(t)dt \right)^{1/q} V^s(x);$$

$$A_2(s) := \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)dt \right)^{1/p'} U^s(x);$$

$$A_3(s) := \sup_{0<x<b} \left(\int_0^x u(t) V^{q(\frac{1}{p'}+s)}(t)dt \right)^{1/q} V^{-s}(x);$$

$$A_4(s) := \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt \right)^{1/p'} U^{-s}(x).$$

Remark 3.14. **Note that**

$$A_{MB} = A_1\left(\frac{1}{p'}\right), \quad A_{PS} = A_3\left(\frac{1}{p}\right).$$

Also all other known alternative conditions are just points on these cases.

Remark 3.15. **A similar result for the case $1 < q < p < \infty$ is also known and proved by L.E. Persson with V. Stepanov.**

3.6 More on multi-dimensional Hardy-type inequalities

In this Section by a weight we mean a non-negative, measurable and locally integrable function on $\mathbb{R}^n_+,\ n \in \mathbb{Z}.$

The main information in this section can be found in recent papers by L.E.Persson and his students A.Wedestig (PhD 2004) and E.Ushakova (PhD 2006).

We refer to the book [C] and the review article [$*$] by L.E.Persson and N.Samko from 2010, where also complementary information can be found.

[$*$] L.E.Persson and N.Samko. Some remarks and new developments concerning Hardy-type inequalities. Rend. Circ. Mat. Palermo, Suppl. 82(2010), 93-122.

Some two-dimensional results

We first recall that the following two-dimensional inequality, which was proved by E.T. Sawyer in 1985:

Theorem 3.16. ***Let $1 < p \le q < \infty$ and u and v be weights on $\mathbb{R}^2_+$. Then the inequality***

$$\left(\int\limits_0^\infty\int\limits_0^\infty\left(\int\limits_0^{x_1}\int\limits_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{q}} \le$$

$$\le C\left(\int\limits_0^\infty\int\limits_0^\infty f^p(x_1,x_2)v(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{p}} \tag{3.16}$$

holds for all non-negative and measurable functions on $\mathbb{R}^2_+$, if and only if the following three conditions are satisfied:

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\left(\int\limits_{y_1}^\infty\int\limits_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\left(\int\limits_0^{y_1}\int\limits_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p'}} < \infty, \tag{3.17}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\frac{\left(\int\limits_0^{y_1}\int\limits_0^{y_2}\left(\int\limits_0^{x_1}\int\limits_0^{x_2} v(t_1,t_2)^{1-p'}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int\limits_0^{y_1}\int\limits_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p}}} < \infty, \tag{3.18}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\frac{\left(\int\limits_{y_1}^\infty\int\limits_{y_2}^\infty\left(\int\limits_{x_1}^\infty\int\limits_{x_2}^\infty u(t_1,t_2)dt_1dt_2\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int\limits_{y_1}^\infty\int\limits_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q'}}} < \infty. \tag{3.19}$$

All three conditions (3.17)**-**(3.19) **are independent and no one may be removed.**

Remark 3.17. **Note that** (3.17) **corresponds to the Muckenhoupt-Bradley condition** (2.2)**,** (3.21) **corresponds to the condition** (2.3) **and** (3.19) **corresponds to the dual condition of** (2.3)**. According to Theorem 3.13 and Remark 3.14 all these conditions are equivalent in the one-dimensional case but it is not so in the two-dimensional case.**

One of the recent progresses related to Theorem 3.16 was obtained in A. Wedestig's PhD thesis from 2004. It was shown there that in the case where the weight $v(x_1, x_2)$ **on the right-hand side of** (3.16) **has the form of the product** $v_1(x_1)v_2(x_2)$**, then only one condition appears (but this condition is not unique and can in fact be given in infinite many forms).**

Namely, the following statement holds:

Theorem 3.18. ***Let*** $1 < p \le q < \infty$ ***and let*** u ***be a weight on*** $\mathbb{R}^2_+$ ***and*** v_1 ***and*** v_2 ***be weights on*** $\mathbb{R}_+$. ***Then the inequality***

$$\left(\int_0^\infty \int_0^\infty \left(\int_0^{x_1} \int_0^{x_2} f(t_1, t_2) dt_1 dt_2 \right)^q u(x_1, x_2)\, dx_1 dx_2 \right)^{\frac{1}{q}} \le$$

$$\le C \left(\int_0^\infty \int_0^\infty f^p(x_1, x_2) v_1(x_1) v_2(x_2)\, dx_1 dx_2 \right)^{\frac{1}{p}} \tag{3.20}$$

holds for all non-negative and measurable functions f ***on*** $\mathbb{R}^2_+$, ***if and only if***

$$A_W(s_1, s_2) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} (V_1(t_1))^{\frac{s_1-1}{p}} (V_2(t_2))^{\frac{s_2-1}{p}} \times$$

$$\left(\int_{t_1}^\infty \int_{t_2}^\infty u(x_1, x_2)(V_1(x_1))^{q\frac{p-s_1}{p}} (V_2(x_2))^{q\frac{p-s_2}{p}}\, dx_1 dx_2 \right)^{\frac{1}{q}} < \infty$$

holds for some $s_1, s_2 \in (1, p)$ ***(and, hence, for all*** $s_1, s_2 \in (1, p)$***), where*** $V_i(t_i) := \int_0^{t_i} v_i(\xi)^{1-p'} d\xi,\ i = 1, 2$. ***Moreover, for the best constant*** C ***in*** (3.20) ***it yields that*** $C \approx A_W(s_1, s_2)$.

A limit result of Theorem 3.18 is the following two-dimensional Pólya-Knopp type inequality, which was also proved in the same PhD thesis:

Theorem 3.19. *Let* $0 < p \le q < \infty$ *and* u *and* v *be weights on* $\mathbb{R}^2_+$*. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(t_1,t_2)dt_1dt_2\right)\right]^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}$$

$$\le C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \qquad (3.21)$$

holds for all non-negative and measurable functions f *on* $\mathbb{R}^2_+$ *if and only if*

$$\sup_{y_1>0,y_2>0} y_1^{\frac{s_1-1}{p}}y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^\infty\int_{y_2}^\infty x_1^{-\frac{s_1q}{p}}x_2^{-\frac{s_2q}{p}}w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} < \infty,$$

holds for some $s_1 > 1, s_2 > 1$ *(and thus for all* $s_1 > 1, s_2 > 1$ *) and where*

$$w(x_1,x_2) := u(x_1,x_2)\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_1dt_2\right)\right]^{\frac{q}{p}}.$$

Remark 3.20. **Observe that this limit inequality indeed holds for all weights (and not only for product weights on the right hand side) and also for $0 < p \leq 1.$ The reason for this comes from the useful technical details when we perform the limit procedure, e.g. that we first do a substitution so we only need to use the case when the weight in the right hand side in (3.20) is equal to 1. Also here we have a good estimate of the best constant C in (3.21).**

Remark 3.21. **The corresponding statements as those in Theorems 3.18 and 3.19 hold also for any dimension $n.$ However, in our next Subsection we will present some results mainly from the PhD thesis of E.Ushakova from 2006, where also the case with product weights on the left hand side was considered. The proofs there are completely different from those before and the obtained characterizations are different.**

Some more multidimensional results

In the sequel we assume that f is a non-negative and measurable function.

Let $x = (x_1, ..., x_n), t = (t_1, ..., t_n) \in \mathbb{R}^n_+, n \in Z_+$ and $1 < p \leq q < \infty$. We consider the $n-$ dimensional Hardy type operator

$$(H_n f)(x) = \int_0^{x_1} \cdots \int_0^{x_n} f(t)dt$$

and study the inequality

$$\left(\int_{\mathbb{R}^n_+} (H_n f)^q(x) u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n_+} f^p(x) v(x) dx \right)^{\frac{1}{p}}. \tag{3.22}$$

Sometimes we assume that one of the involved weight functions v and u is of product type, i.e. that

$$u(x) = u_1(x_1) u_2(x_2) \cdots u_n(x_n), \qquad (LP)$$

or

$$v(x) = v_1(x_1) v_2(x_2) \cdots v_n(x_n). \qquad (RP)$$

Moreover,

$$U(t) = U(t_1, \cdot \cdot \cdot, t_n) := \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} u(x)dx$$

and

$$V(t) = V(t_1, \cdot \cdot \cdot, t_n) := \int_0^{t_1} \cdots \int_0^{t_n} (v(x))^{1-p'} dx.$$

The next Statement gives a necessary condition for (3.22) **to hold with help of some** $n-$ **dimensional versions of the constants** A_{MB} **and** A_{PS} **in Theorem 2.1.**

Theorem 3.22. ***Let*** $1 < p \leq q < \infty$ ***and assume that*** (3.22) ***holds for all non-negative and measurable functions*** f ***on*** $\mathbb{R}^n_+$ ***with a finite constant*** C***, which is independent on*** f***. Then***

$$A^{(n)}_{MB} := \sup_{t_i > 0} (U(t_1, \cdots, t_n))^{1/q} (V(t_1, \cdots, t_n))^{1/p\prime} < \infty,$$

and

$$A^{(n)}_{PS} := \sup_{t_i > o} \left(V(t_1, \cdots, t_n)\right)^{-1/p} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} u(x) V^q(x) dx \right)^{1/q} < \infty.$$

Our next result is that in the case of product weights on the right hand side we get a complete characterization of (3.22)**.**

Theorem 3.23. ***Let*** $1 < p \leq q < \infty$ ***and the weight*** v ***be of product type*** (RP). ***Then*** (3.22) ***holds for all non-negative and measurable functions*** f ***on*** $\mathbb{R}_+^n$ ***with some finite constant*** C, ***which is independent on*** f, ***if and only if*** $A_{MB}^{(n)} < \infty$ ***or*** $A_{PS}^{(n)} < \infty$. ***Moreover,*** $C \approx A_{MB}^{(n)} \approx A_{PS}^{(n)}$ ***with constants of equivalence only depending on the parameters*** p ***and*** q ***and the dimension*** n***.***

Note that here it yields that $V(t_1, \cdot \cdot \cdot, t_n) = V_1(x_1)V_2(x_2) \cdot \cdot \cdot V_n(x_n)$, **where** $V_i(t_i) := \int_0^{t_i}(v_i(x_i))^{1-p'}dx_i, i = 1, \cdot \cdot \cdot, n$. **For a proof we refer to the mentioned PhD thesis (see also the book [C]).**

We can also consider the case when u **is of product type** (LP) **and where we need the dual of the constants** $A_{MB}^{(n)}$ **and** $A_{PS}^{(n)}$:

$$A_{MB}^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots U_n(t_n)\big)^{1/q} \big(V(t_1, \cdots, t_n)\big)^{1/p\prime} < \infty,$$

and

$$A_{PS}^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots$$

$$\cdots U_n(t_n)\big)^{-1/q'} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} v^{1-p'}(x) \big(U_1(x_1) \cdots U_n(x_n)\big)^{p'} dx\right)^{1/p'}.$$

Theorem 3.24. ***Let*** $1 < p \leq q < \infty$ ***and the weight*** u ***be of product type*** (LP)***. Then*** (3.22) ***holds for all non-negative and measurable functions*** f ***on*** $\mathbb{R}^n_+$ ***with some finite constant*** C***, which is independent of*** f***, if and only if*** $A_M^{*(n)} < \infty$ ***or*** $A_{PS}^{*(n)} < \infty$. ***Moreover,*** $C \approx A_M^{*(n)} \approx A_{PS}^{*(n)}$ ***with constants of equivalence only depending on the parameters*** p ***and*** q ***and the dimension*** n***.***

Also the case $1 < q < p < \infty$ **can be considered and the following multidimensional versions of the usual Mazya-Rosin and Persson-Stepanov constants in one dimension can be defined:**

$$B_{MR}^{(n)} := \left(\int_{\mathbb{R}^n_+} \big(U(t)\big)^{r/q} \big(V_1(t_1)\big)^{r/q'} \cdots \big(V_n(t_n)\big)^{r/q'} dV_1(t_1) \cdots dV_n(t_n) \right)^{1/r},$$

$$B_{PS}^{(n)} := \Bigg(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \cdots \int_0^{t_n} u(x)(V_1(x_1) \cdots V_n(x_n))^q dx \right)^{r/q} \times$$

$$\times \Big(V_1(t_1) \cdots V_n(t_n) \Big)^{-r/q} dV_1(t_1) \cdots dV_n(t_n) \Bigg)^{1/r}.$$

Here, as usual, $1/r = 1/q - 1/p$**. For technical reasons we also need the following additional condition:**

$$V_1(\infty) = \cdots = V_n(\infty) = \infty.$$

Theorem 3.25. *Let* $1 < q < p < \infty$ *and* $1/r = 1/q - 1/p$. *Assume that the weight* v *is of product type* (RP)*. Then* (3.22) *holds for all non-negative and measurable functions* f *on* $\mathbb{R}^n_+$ *with some finite constant* C*, which is independent on* f*, if and only if* $B^{(n)}_{MR} < \infty$, *or* $B^{(n)}_{PS} < \infty$. *Moreover,* $C \approx B^{(n)}_{MR} \approx B^{(n)}_{PS}$ *with constants of equivalence depending only on* p *and* q *and the dimension* n*.*

Remark 3.26. Also for $1 < p < q < \infty$ the case when the left hand side is of product type can be considered and a theorem similar to Theorem 3.25 can be proved by using some dual forms of the constants $B^{(n)}_{MR}$ and $B^{(n)}_{PS}$.

We finalize this Section by shortly discussing some limit multidimensional (Pólya-Knopp type) inequalities. Consider the inequality

$$\left(\int_{\mathbb{R}^n_+} \big(G_n f\big)^q (x) u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n_+} f^p(x) v(x) dx \right)^{1/p}, \tag{3.23}$$

where the $n-$ dimensional geometric mean operator G_n is defined by

$$\big(G_n f\big)(x) = \exp\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} ln f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \right).$$

We denote

$$A_G^{(n)} := \sup_{t_i>0}(t_1 \cdots t_n)^{-1/p} \left(\int_0^{t_1} \cdots \int_0^{t_n} w(x)dx \right)^{1/q}$$

with

$$w(x) := \big((G_n v)(x) \big)^{-q/p} u(x).$$

Theorem 3.27. ***Let*** $0 < p \leq q < \infty$. ***Then*** (3.23) ***holds for all non-negative and measurable functions on*** $\mathbb{R}_+^n$ ***if and only if*** $A_G^{(n)} < \infty$. ***Moreover,*** $C \approx A_G^{(n)}$ ***with constants of equivalence depending only on the parameters*** p ***and*** q ***and the dimension*** n***.***

Remark 3.28. **Our proof shows that Theorem 3.27 may be regarded as a natural limit case of Theorem 3.23 characterized by the condition** $A_{PS}^{(n)} < \infty$. **For** $n = 2$ **we get another characterization than that in Theorem 3.19. Note that also in this case the limit result holds in a wider range of parameters and for general weights.**

Remark 3.29. **A similar result can be derived also for the case** $0 < q < p < \infty$ **now as a limiting case of Theorem 3.25.**

Thank you for your attention!

Yours Lars-Erik